

# ETA AND RHO INVARIANTS ON MANIFOLDS WITH EDGES

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**ABSTRACT.** We establish existence of the eta-invariant as well as of the Atiyah-Patodi-Singer and the Cheeger-Gromov rho-invariants for a class of Dirac operators on an incomplete edge space. Our analysis applies in particular to the signature, the Gauss-Bonnet and the spin Dirac operator. We derive an analogue of the Atiyah-Patodi-Singer index theorem for incomplete edge spaces and their non-compact infinite Galois coverings with edge singular boundary. Our arguments employ microlocal analysis of the heat kernel asymptotics on incomplete edge spaces and the classical argument of Atiyah-Patodi-Singer. As an application, we discuss stability results for the two rho-invariants we have defined.

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## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

**1.1. Eta function and eta invariant.** Consider a compact Riemannian manifold  $(M, g)$  with  $\dim M = m$  and assume for the moment that  $M$  is closed and smooth, i.e. without singularities. Consider a Hermitian vector bundle  $(E, h)$  over  $M$  and denote the space of smooth sections over  $M$  with values in the vector bundle  $E$  by  $\Gamma(M, E)$ . Let  $D$  be a linear elliptic self-adjoint first order differential operator  $D$  acting on the smooth sections  $\Gamma(M, E)$ . The eta invariant of  $D$  was introduced by Atiyah, Patodi and Singer [APS75A], see also [APS75B, APS76], to measure the spectral asymmetry of

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$D$  in the following way. Consider an enumeration  $\{\lambda_n\}_{n \in \mathbb{N}_0}$  of the non-zero eigenvalues of  $D$ , counted with their multiplicities and ordered e.g. in ascending order. Denote by  $\text{sign}(\lambda)$  the sign of an eigenvalue  $\lambda$ . Then the eta function of  $D$  is defined by

$$\eta(D, s) := \sum_{n=0}^{\infty} \text{sign}(\lambda_n) |\lambda_n|^{-s}, \quad \text{Re}(s) > m.$$

This series is absolutely convergent for  $\text{Re}(s) > m$  and, as a consequence of the short time asymptotics of the trace  $\text{Tr } D e^{-tD^2}$ , extends to a meromorphic function on the whole of  $\mathbb{C}$  by the following integral expression

$$(1.1) \quad \eta(D, s) = \frac{1}{\Gamma((s+1)/2)} \int_0^{\infty} t^{(s-1)/2} \text{Tr } D e^{-tD^2} dt$$

Regularity of the eta function at  $s = 0$  is a general phenomenon that has been established by Atiyah, Patodi and Singer [APS75A] by viewing  $M$  as boundary of a half cylinder  $M \times \mathbb{R}^+$  and studying the Dirac-type operator  $\mathbb{D} := (\partial_u + D)$  on the cylinder with Atiyah-Patodi-Singer boundary conditions at  $u = 0$ . Here,  $u \in \mathbb{R}^+$  denotes the variable along the cylinder. If the eta-function is regular at  $s = 0$ , its value at zero is defined as the eta-invariant of  $D$ .

Under additional assumptions on the operator  $D$ , namely  $D$  is a Dirac operator associated to a unitary Clifford action and to a Clifford connection, one can prove that the right hand side of (1.1), evaluated at  $s = 0$ , namely

$$\frac{1}{\sqrt{\pi}} \int_0^{\infty} \text{Tr } D e^{-tD^2} \frac{dt}{\sqrt{t}},$$

converges (see [BiFr86]) and this value is the eta invariant of  $D$ . We point out that if  $M$  is even dimensional and  $D$  is an odd  $\mathbb{Z}_2$ -graded Dirac-type operator acting on the sections of  $E = E^+ \oplus E^-$ , then  $D e^{-tD^2}$  is an odd operator and thus its trace vanishes identically for all positive times

$$\text{Tr } D e^{-tD^2} \equiv 0.$$

Equivalently, one can prove, using for example the isomorphism  $U : L^2(M, E^+) \rightarrow L^2(M, E^-)$  induced by Clifford multiplication by  $s \in L^2(M, T^*M)$  with pointwise norm  $|s(p)| = 1$  almost everywhere in  $p \in M$ , that the spectrum of such an operator  $D$  is symmetric around zero, and thus its eta invariant vanishes.

However, the spectral symmetry may be broken by adding a potential, which may lead to a non-trivial eta-invariant in even dimensions. Another geometric example of a non-trivial eta invariant in even dimensions is given by Gilkey [Gil85].

The eta function has been studied extensively in various settings, for instance by Bismut and Freed [BiFr86], Branson and Gilkey [BRGi92], Botvinnik and Gilkey [BoGi97], [BoGi95], Gilkey [Gil81], Goette [GOE99, GOE00, GOE09]. In case of manifolds with boundary, we mention contributions by Bunke [BUN95], Gilkey and Smith [GiSm83], Lesch and Wojciechowski [LEW096], Müller [MUE94] and Melrose [MEL93]. In the setting of isolated conical singularities we refer the reader to the seminal work by Cheeger

[CHE83, CHE87] and Lesch [LES97]. We also refer the reader to an in-depth overview article by Goette [GOE11] and the references therein.

**1.2. Atiyah-Patodi-Singer index theorem.** The eta invariant  $\eta(D)$  appears as the correction term in the Atiyah-Patodi-Singer index theorem on manifolds with boundary. More precisely, consider an even dimensional compact Riemannian manifold  $(X, G)$  with boundary  $(M, g)$  and a linear elliptic first order differential operator  $\mathbb{D}$  acting between the sections of two Hermitian vector bundles  $E$  and  $F$ . Assume that over the collar  $M \times [0, \varepsilon)$  of the boundary,  $\mathbb{D}$  takes a special form

$$(1.2) \quad \mathbb{D} = \sigma \left( \frac{\partial}{\partial u} + D \right),$$

where  $u \in [0, \varepsilon)$  is the inward normal coordinate and  $\sigma$  is a bundle isomorphism  $E \upharpoonright M \rightarrow F \upharpoonright M$ . The tangential operator  $D$  is a self-adjoint operator acting on sections of  $E \upharpoonright M$ . Impose Atiyah-Patodi-Singer boundary conditions at  $M$ , defined in terms of the positive spectral projection of  $D$ . Then  $\mathbb{D}$  is Fredholm and for its index,  $\text{index } \mathbb{D} = \dim \ker \mathbb{D} - \dim \ker \mathbb{D}^*$ , the following remarkable formula holds:

$$(1.3) \quad \text{index } \mathbb{D} = \int_X a_0(p) \, \text{dvol}_G(p) - \frac{\dim \ker D + \eta(D)}{2}.$$

Here  $a_0(p)$  is obtained as the constant term in the short time asymptotic expansion of the pointwise trace of  $\exp(-t\mathbb{D}^*\mathbb{D}) - \exp(-t\mathbb{D}\mathbb{D}^*)$  at  $p \in M$ ; it is the same coefficient that would appear in the boundaryless case. If  $\mathbb{D}$  is the positive part of an odd Dirac-type operator acting between the sections of a  $\mathbb{Z}_2$ -graded Clifford module, then  $a_0$  can be explicitly written down in terms of the curvature tensor of  $M$  and of  $E$ . For example, in case  $\mathbb{D}$  is the spin Dirac operator acting between the sections of the positive and the negative spinor bundles, then  $a_0$  can be identified as the  $\hat{A}$ -polynomial applied to the curvature tensor of  $X$ .

**1.3. Atiyah-Patodi-Singer index theorem for Galois coverings.** The Atiyah-Patodi-Singer index theorem has a non-compact analogue on infinite Galois coverings, a result due to Ramachandran [RAM93]; this result is a generalisation to manifolds with boundary of the seminal result by Atiyah [AT76]. Given a Galois covering  $\tilde{X}$  of the manifold  $X$  with Galois group  $\Gamma$ , one may lift the Dirac operator  $\mathbb{D}$  on  $X$  to a  $\Gamma$ -invariant operator  $\tilde{\mathbb{D}}$  on the covering and show that there is a well-defined  $\Gamma$ -index  $\text{index}_\Gamma$  for a suitable generalization of the Atiyah-Patodi-Singer boundary conditions. The boundary operator of  $\tilde{\mathbb{D}}$  is denoted by  $\tilde{D}$ ; it is a  $\Gamma$ -invariant operator on  $\partial\tilde{X}$  descending to  $D$  on  $\partial X$ . Ramachandran proves existence of a  $\Gamma$ -eta invariant  $\eta_\Gamma(\tilde{D})$  and establishes the index formula

$$(1.4) \quad \text{index}_\Gamma \tilde{\mathbb{D}} = \int_X a_0(p) \, \text{dvol}_G(p) - \frac{\dim_\Gamma \ker \tilde{D} + \eta_\Gamma(\tilde{D})}{2},$$

where  $a_0$  is again the usual coefficient from above. Both the  $\Gamma$ -index, the  $\Gamma$ -dimension of  $\ker \tilde{D}$  and the  $\Gamma$ -eta invariant make use of a trace on a suitable Von Neumann algebra.

**1.4. Cheeger-Gromov and Atiyah-Patodi-Singer rho-invariants.** While the eta invariant has a prominent role as the correction term in the index formula on manifolds with boundary, it can in fact be used to study geometric questions on closed manifolds independently. Particularly interesting are the Cheeger-Gromov and the Atiyah-Patodi-Singer rho invariants, defined in terms of eta invariants in the following way. The Cheeger-Gromov rho invariant is defined by

$$(1.5) \quad \rho_{\Gamma}(D) := \eta_{\Gamma}(\tilde{D}) - \eta(D).$$

The Atiyah-Patodi-Singer rho invariant is associated to a pair of unitary representations of the fundamental group  $\pi_1(M)$  of the same dimension:  $\alpha, \beta : \pi_1(M) \rightarrow U(\ell)$ . Consider the flat vector bundles

$$E_{\alpha} = \tilde{M} \times_{\alpha} \mathbb{C}^{\ell}, \quad E_{\beta} = \tilde{M} \times_{\beta} \mathbb{C}^{\ell}$$

We can define twisted Dirac operators  $D_{\alpha}$  and  $D_{\beta}$ , respectively. One then defines the Atiyah-Patodi-Singer rho invariant associated to  $(D, \alpha, \beta)$  by

$$(1.6) \quad \rho_{\alpha-\beta}(D) := \eta(D_{\alpha}) - \eta(D_{\beta}).$$

Rho invariants have very interesting applications. For example, the spin Dirac operators can be used in order to distinguish bordism classes of metrics of metrics of positive scalar curvature; this is work of Botvinnik and Gilkey [BoGi95] and, more generally, Piazza-Schick [PiSh07A]. Similarly, rho invariants for the signature operator are employed in order to distinguish homotopy equivalent manifolds that are non-diffeomorphic; this is work of Chang-Weinberger [CHWe03].

All these contributions assume that there is some torsion in the fundamental group of the manifold. If, on the other hand, the fundamental group of the manifold is torsion free and the (maximal) Baum-Connes map is an isomorphism, then these secondary invariants behave like primary invariants, i.e. they are zero for the spin Dirac operator associated to a positive scalar curvature metric and they are homotopy invariant for the signature operator. This is a theorem of Keswani [Kes00A, Kes00B], reproved by Piazza-Schick using bordism techniques, see [PiSh07B], and reproved lately by Higson-Roe [HiRo10] for the APS-rho invariant and by Benameur-Roy [BeRo14] for the Cheeger-Gromov rho invariant, using coarse index theory techniques.

**1.5. Statement of the main results.** In this paper we study the classical statements from above in case where the compact manifold  $M$  admits an incomplete edge singularity. Index theory has been extended from manifolds with boundary to spaces with conical singularities by Cheeger e.g. in [CHE83], see also Brüning and Seeley [BRSe88], Lesch [LES97], Lesch and Wojciechowski [LEWo96], Fedosov-Schulze-Tarkhanov [FST99] and Chou [CHO85], to name a few. Singular analysis has been employed by Bismut and Cheeger [BiCh90] in their families index theorem on manifolds with boundary, where they assumed invertibility of the boundary Dirac operators. Let us also mention work by Brüning [BRU09] for the signature operator on simple edge spaces of Witt type and more recently by Albin and Gell-Redman [ALGE13] for the spin Dirac operator on simple edge spaces satisfying a geometric Witt condition; in both these articles an explicit index formula is proved.

Hodge theory on singular spaces has been developed by Cheeger in his seminal papers e.g. [CHE83] and by Lesch [LES97], Hunzicker-Mazzeo [HuMA05], Bei [Bei12], Albin, Leichtnam, Mazzeo and the first named author in [ALMP12], as well as in [ALMP13].

A simple incomplete edge manifold is a smoothly stratified space of depth equal to one. Such a space admits a resolution given by a compact manifold  $M_c$  with a boundary  $\partial M_c$  which fibers  $\phi : \partial M_c \rightarrow B$  over a closed manifold  $B$ , the edge singularity, and fibre  $F$ , a closed manifold as well. A precise definition of smoothly stratified spaces of any depth is given in [BHS91], cf. also [ALMP15]. Given a Riemannian metric  $g^B$  on the edge  $B$ , and a symmetric 2-tensor  $\kappa$  on  $\partial M$  that restricts to an isospectral family of Riemannian metrics on the fibres, the singular edge structure in a neighborhood  $\mathcal{U}$  of the boundary is given by the Riemannian metric ( $x$  denotes the defining function of the boundary)

$$g \upharpoonright \mathcal{U} = dx^2 + x^2 \kappa + \phi^* g^B.$$

In this paper we shall mainly concentrate on the depth-1 case, thus requiring the links  $F$  to be compact and smooth. However, for later use, we shall also allow special depth-2 stratified spaces, with the most singular stratum equal to a discrete collection of points  $P$  and associated link equal to a depth-1 stratified space; we shall also require the metric in a neighbourhood  $\mathcal{U}$  of  $P$  to be exact, i.e. of the form  $g \upharpoonright \mathcal{U} = dx^2 + x^2 \kappa$ . In this specific situation, our analysis actually allows for the link  $F$  to be any compact smoothly stratified space, as long as self-adjoint extensions of certain geometric operators on  $F$  admit discrete spectrum of finite multiplicity.

Our arguments apply to any formally self-adjoint Dirac-type operator  $D$  on  $M$  that it is equal to one of three geometric Dirac operators, namely the Gauss-Bonnet, the signature and the spin Dirac operators, modulo lower order terms. We shall be more precise later; for the time being we call such Dirac-type operators *allowable* (while we keep the word *geometric* for the unperturbed operators<sup>1</sup>). Moreover we require  $D$  to be even or odd with respect to some even subcalculus of the heat calculus (see Section 4 for the relevant notions).

Our first main result discusses existence of eta invariants for such operators under certain additional assumptions on the metric  $g$  and under a geometric Witt condition on  $D$  which will be made explicit below, see Assumption 2.4.

**Theorem 1.1.** *Assume  $(M, g)$  is an incomplete simple edge space with an admissible edge metric. Let  $D$  be an allowable Dirac operator satisfying the geometric Witt condition; let  $m$  denote the dimension of  $M$  and  $b$  the dimension of the edge  $B$ . Then we have:*

- (i) *Assume  $m$  is even and  $(D, b)$  are either both of even or both of odd parity. Then the eta invariant  $\eta(D)$  is well-defined.*
- (ii) *Assume  $m$  is odd, and  $(D, b)$  are either both of even or both of odd parity. Then the residue of  $\eta(D, s)$  at  $s = 0$  comes from the interior only.*

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<sup>1</sup>In this paper we shall care to distinguish the signature operator in even and odd dimensions and for this reason we shall often refer to them as the signature operator (in even dimensions) and the odd-signature operator (in odd dimensions).

- (iii) Assume  $D$  is even and either  $(m-b)$  is odd or  $b$  is even; or  $D$  is odd and either  $(m-b)$  is even or  $b$  is odd. Then the residue of  $\eta(D, s)$  at  $s = 0$  comes from the interior and from the edge fibration.
- (iv) Assume  $D$  is even,  $(m-b)$  is even and  $b$  is odd; or if  $D$  is odd,  $(m-b)$  is odd and  $b$  is even. Then the eta function  $\eta(D, s)$  may admit a second order pole singularity at  $s = 0$ .
- (v) Assume  $M$  is boundary of some admissible edge space  $X$  with an even or odd Dirac operator  $\mathbb{D}$ , satisfying the geometric Witt condition and such that  $D$  is the tangential operator of  $\mathbb{D}$  as in (1.2). Moreover, assume that for the dimension  $b_i$  of each edge in  $X$ , either  $(m+1-b_i)$  or  $b_i$  is odd. Then the eta-invariant  $\eta(D)$  is well defined.

Corresponding statements hold also for the  $\Gamma$ -eta invariant  $\eta_\Gamma(D)$  on Galois coverings of admissible edge spaces.

The special case of *geometric* Dirac operators, i.e., once again, the Gauss Bonnet, the signature and the spin Dirac operators, is included in the general theorem above. As already remarked, the eta-invariants of these operators are trivially zero in case of dimension  $m$  being even, due to symmetry of the spectrum. However, when discussing general *allowable* Dirac operators, the geometric examples may well be perturbed in such a way to destroy the spectral symmetry.

A remark on Theorem 1.2 (v) is in order. Let  $D$  be equal to the signature operator and let  $M$  be odd dimensional with an edge  $B$  of even dimension. We can always construct a bounding stratified pseudomanifold  $X$  by setting  $X$  to be a finite cone over  $M$ :  $X = \mathcal{C}(M)$ . This defines an iterated cone edge space which may still be covered by the analytic arguments given here (we will still refer to Theorem 1.2 (v), without mentioning this extension). If now  $M$  is odd dimensional with even dimensional edges and satisfies the geometric Witt condition, the cone over  $M$  also satisfies the geometric Witt condition (indeed, the link of the tip of the cone is odd dimensional) and so all the assumptions of Theorem 1.2 (v) are fulfilled and we can conclude that the eta invariant of the odd-signature operator on  $M$  is well defined. The precise statement is given in Corollary 8.3. Similarly, if  $M$  is spin, odd dimensional with an edge  $B$  of even dimension and an edge metric of uniform positive scalar curvature then the cone over  $M$  satisfies the geometric Witt condition (provided we suitably scale the metric) and so, by Theorem 1.2 (v), the eta invariant of the spin Dirac operator on  $M$  is well defined. These two examples are particularly important for applications and therefore singled out in Corollary 8.3.

A central component in our argument is the even/odd heat calculus developed by the second author jointly with Mazzeo in [MAVE12]. In that reference the authors studied analytic torsion on simple edge manifolds, which required a detailed analysis of the heat kernel. In the present paper we give more details and provide a more direct treatment of the even and odd subcalculi. The heat kernel on a manifold with isolated conical singularities has been studied by Cheeger in his seminal paper [CHE83]; the general Witt case is also treated in [CHE83], see the iterative argument given in Section 7.4



there<sup>2</sup>. Mooers [Moo99] gives a microlocal approach to the study of the heat kernel in the isolated case; this is the approach that is developed by Mazzeo and the second author.

Going back to the contents of this paper, our second main result extends the Atiyah-Patodi-Singer index formula on compact manifolds with boundary  $M$  and on their infinite Galois coverings to the case of  $M$  being an edge manifold as above.

**Theorem 1.2.** *Consider an incomplete edge space  $M$  with an admissible edge metric  $g$ . Let  $D$  be an allowable Dirac operator, satisfying the geometric Witt condition. Assume  $M$  is boundary of some incomplete edge space  $X$  with an admissible edge metric and an allowable Dirac operator  $\mathbb{D}$  of the form (1.2) near the boundary, satisfying the geometric Witt condition. We denote by  $Y$  the fibration of links over the singular stratum of  $X$ . Assume that  $\mathbb{D}^2$  is even and that for the dimension  $b_i$  of each edge singularity in  $\tilde{X}$  either  $(m+1-b_i)$  or  $b_i$  is odd. Then the Dirac operator  $\mathbb{D}$  with APS boundary conditions is Fredholm, the eta invariant of  $D$  is well-defined and is related to the Fredholm index of  $\mathbb{D}$  by the index formula*

$$(1.7) \quad \text{index } \mathbb{D} = \left( \int_X a_0 + \int_Y b_0 \right) - \frac{\dim \ker D + \eta(D)}{2},$$

where we have the following characterization of the integrands  $a_0$  and  $b_0$ .

- (i) *The integrand  $a_0$  comes from the interior in the sense that at each  $p \in X$  the value  $a_0(p)$  depends only on the jets of the Riemannian metric and the coefficients of  $\mathbb{D}$  at  $p$ . The function  $a_0$  is in fact the same as in the classical formula (1.3).*
- (ii) *The integrand  $b_0$  comes from the edge singularity in the sense that at each  $p \in Y$  the value  $b_0(p)$  depends only on the jets of the Riemannian metric and the coefficients of  $\mathbb{D}$  at  $p$ .*
- (iii) *If  $\mathbb{D}$  is an allowable perturbation of a geometric Dirac operator twisted by a flat vector bundle  $E$ , then the coefficients  $a_0$  and  $b_0$  depend only on the rank of  $E$ .*

*The corresponding statement carries over to the setting of Galois coverings, where index of  $\mathbb{D}$ , dimension of  $\ker D$  and the eta-invariant  $\eta(D)$  are replaced by their corresponding Galois covering versions.*

**Remark 1.3.** We point out that the presence of an integral along the edge singularity in the index formulae (1.7) associated to an edge manifold is a general phenomenon and in fact appears already in the signature index formula of Brüning [BRU09] and in the spin index formula of Albin and Gell-Redman [ALGE13]. In fact one is tempted to identify the integrands  $a_0$  and  $b_0$  with the integrands on the right hand sides of the

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<sup>2</sup>Cheeger [CHE83, §7.4], studies the heat kernel of the signature operator on Witt spaces of arbitrary depth. In that respect his result is much more general than the Witt spaces of depth one, considered here. However, it should be remarked that Cheeger considers piecewise flat cone-edge metrics, which are locally isometric to products of smooth open subsets with exact cones. Thus the edge metrics considered here are much more general, since we allow for fibrations of not necessarily exact cones and moreover do not require the fibrations to be locally isometric to products of smooth open subsets and cones.

formulae in [BRU09, Theorem 0-2] and [ALGE13, (4)] in the signature and the spin Dirac case. However, the results in [BRU09] and [ALGE13] do not encompass a local index formula and hence a direct identification of  $a_0$  and  $b_0$  is not straightforward.

Next we prove the important result that the rho invariants of Atiyah-Patodi-Singer and of Cheeger-Gromov are well defined in the singular setting.

**Theorem 1.4.** *Assume  $(M, g)$  is an incomplete simple edge space with an admissible edge metric. Let  $D$  be an allowable Dirac operator satisfying the geometric Witt condition. Then the APS and the Cheeger-Gromov rho invariants are well-defined.*

As a consequence of Theorem 1.2 we conclude the paper with the following stability results.

**Theorem 1.5.** *Assume  $(M, g)$  is an incomplete simple edge space with an admissible edge metric. The following holds:*

- (i) *the APS and Cheeger-Gromov rho invariants of the spin Dirac operator associated to an edge metric of positive scalar curvature are bordism invariant of positive scalar curvature edge metrics;*
- (ii) *the APS and Cheeger-Gromov rho invariants for the signature operator are invariant under variations of  $g$  among admissible edge metrics satisfying the Witt condition.*

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## 2. REVIEW OF GEOMETRY ON INCOMPLETE EDGE SPACES

Consider a compact smoothly stratified space  $\overline{M}$  of depth 1 and dimension  $m$ : we shall assume that  $\overline{M}$  is the disjoint union of the top stratum  $M$ , a smooth manifold of dimension  $m$ , and a single lower dimensional stratum  $B$ , which is a closed compact manifold of dimension  $b$ . The stratification hypothesis asserts the existence of an open neighbourhood  $U \subset \overline{M}$  around the singular stratum  $B$ , together with a distance function  $x : U \rightarrow [0, 1)$ , such that  $U \cap M$  is the total space of a smooth fibre bundle over  $B$  with the fibre given by a truncated cone  $\mathcal{C}(F) = (0, 1) \times F$  over a compact smooth manifold  $F$  of dimension  $f \geq 1$ . The distance function  $x$  restricts to a radial function of that cone on each fibre. We shall refer to such a stratified space as a *simple edge space*, or, shortly, as an edge space.

The singular stratum  $B$  in  $\overline{M}$  may be resolved to define a compact manifold  $M_c$ , with boundary  $\partial M_c$  being the total space of a fibration  $\phi : \partial M_c \rightarrow B$  with fibre  $F$ . Under the resolution, the neighborhood  $U$  lifts to a collar neighborhood  $\mathcal{U} \subset M_c$ , which is a smooth fibration of cylinders  $[0, 1) \times F$  over  $B$  with radial function  $x$ . The open regular stratum  $M$  is identified with  $M_c \setminus \partial M_c$ .

**Notation:** with a small abuse of notation we shall denote  $\partial M_c$  by  $\partial M$ .

An edge structure on  $M$  is defined by a particular choice of a Riemannian metric.



**Definition 2.1.** A Riemannian simple edge space is a depth-one smoothly stratified space  $\overline{M}$  together with a Riemannian metric  $g$  on  $M = M_c \setminus \partial M$  such that over  $\mathcal{U} \setminus \partial M$  the metric attains the form  $g_0 + h$  with

$$g_0 = dx^2 + x^2\kappa + \phi^*g^B,$$

where  $g^B$  is a Riemannian metric on the closed manifold  $B$ ,  $\kappa$  is a symmetric 2-tensor on the fibration  $\partial M$  restricting to an isospectral family of Riemannian metrics on fibres  $F$ ,  $|h|_{g_0}$  is smooth on  $\mathcal{U}$  and vanishes at  $x = 0$ .

We extend the class of edge singular manifolds slightly in §2.2.

Our analysis requires  $\phi : (\partial M, \kappa + \phi^*g^B) \rightarrow (B, g^B)$  to be a Riemannian submersion in the following sense. If  $p \in \partial M$ , then the tangent bundle  $T_p\partial M$  splits into vertical and horizontal subspaces  $T_p^V\partial M \oplus T_p^H\partial M$ ; the vertical subspace  $T_p^V\partial M$  is the tangent space to the fibre of  $\phi$  through  $p$ , and the horizontal subspace  $T_p^H\partial M$  is the annihilator of the subbundle  $T_p^V\partial M \lrcorner \kappa \subset T^*\partial M$  ( $\lrcorner$  denotes contraction).  $\phi$  is a Riemannian submersion if the tensor  $\kappa$  restricted to  $T_p^H\partial M$  vanishes.

Finally, we require the tensor  $h$  to be even, in the sense that  $h$  admits an asymptotic expansion as  $x \rightarrow 0$  containing only even powers of  $x$ , up to  $dx$ -cross terms which are required to admit only odd powers of  $x$  in their expansion. This condition was introduced in [MAVE12] in order to achieve that the heat kernel of the corresponding Hodge Laplacian lies in a distinguished *even* subcalculus. In fact similar evenness conditions have been employed in various geometric settings, cf. for example [Sco10], [ALB07] and [AOLRo09].

Evenness of  $h$  is well-defined only within a particular *even* equivalence class of coordinate charts near the edge. A local coordinate system  $(\tilde{x}, \tilde{y}, \tilde{z})$  in  $\mathcal{U}$  is said to be in the even equivalence class of a coordinate chart  $(x, y, z)$  if the asymptotics of  $\tilde{x}/x$ ,  $\tilde{y}$  and  $\tilde{z}$  near the edge admits only powers of  $x^2$ , with coefficients in the expansions depending smoothly on  $y$  and  $z$ . We refer to coordinates within a fixed even equivalence class as *special* coordinates and will stay within the special coordinates henceforth.

We summarize these conditions into the notion of *admissible* edge metrics.

**Definition 2.2.** Let  $(M, g)$  be a Riemannian manifold with an edge. The edge metric  $g = g_0 + h$  is said to be admissible if  $\phi : (\partial M, \kappa + \phi^*g^B) \rightarrow (B, g^B)$  is a Riemannian submersion with isospectral fibres  $F$ , and  $h$  is even within a fixed choice of special coordinates in the sense above.

The condition on  $\phi$  to be a Riemannian submersion can be motivated here. Pick local coordinates  $y = (y_1, \dots, y_b)$  on  $B$  lifted to  $\partial M$  and then extended inwards to  $\mathcal{U}$ . Let  $z = (z_1, \dots, z_f)$  restrict to local coordinates on  $F$  along each fibre of the boundary. Such a choice of  $(x, y, z)$  defines local coordinates on  $\mathcal{U} \cap M$ . Consider the Hodge Laplace operator  $\Delta$  on  $(M, g)$  and define for any  $y_0 \in B$  the normal operator  $N(x^2\Delta)_{y_0}$  as the limit of  $x^2\Delta$  with respect to the local family of dilatations  $(x, y, z) \rightarrow (\lambda x, \lambda(y - y_0), z)$  as  $\lambda \rightarrow \infty$ .

If  $\phi$  is a Riemannian submersion,  $N(x^2\Delta_p)_{y_0}$  is naturally identified with  $s^2$  times the Hodge Laplacian on the model edge  $\mathbb{R}_s^+ \times F \times \mathbb{R}^b$  with the model edge metric

$g_{ie} = ds^2 + s^2\kappa_{y_0} + g_{y_0}^B$ , where we identified  $T_{y_0}B = \mathbb{R}^b$ . Similar decomposition holds for the spin Dirac and the spin Laplace operators as worked out in [ALGE13, Lemma 1.1]. This structure of the normal operator is central to the microlocal construction of the heat kernel in [MAVE12], as it allows a decomposition of the initial heat parametrix into an explicit conical and euclidean part.

**2.1. Incomplete edge differential operators.** A central element in our approach to singular edge spaces is the notion of edge vector fields. Define the space of edge vector fields  $\mathcal{V}_e$  to be the space of vector fields smooth in the interior of  $M_c$  and tangent at the boundary  $\partial M_c$  to the fibres of the fibration.  $\mathcal{V}_e$  is closed under the ordinary Lie bracket of vector fields, and hence defines a Lie algebra. In local coordinates, the edge vector fields  $\mathcal{V}_e$  are locally generated by

$$\left\{ x \frac{\partial}{\partial x}, x \frac{\partial}{\partial y_1}, \dots, x \frac{\partial}{\partial y_b}, \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_f} \right\}.$$

The tangent bundle  ${}^eT M$  is defined by asking the edge vector fields  $\mathcal{V}_e$  to form a spanning set of sections, so that  $\mathcal{V}_e = \Gamma({}^eT M)$ . The edge cotangent bundle  ${}^eT^*M$  and is spanned locally the one-forms

$$(2.1) \quad \left\{ \frac{dx}{x}, \frac{dy_1}{x}, \dots, \frac{dy_b}{x}, dz_1, \dots, dz_f \right\}.$$

Though singular in the usual sense, these one-forms are smooth as sections of  ${}^eT^*M$ . We define the incomplete edge cotangent space  ${}^{ie}T^*M$  by setting  $\Gamma({}^{ie}T^*M) = x\Gamma({}^eT^*M)$ , which amounts to  ${}^{ie}T^*M$  being spanned locally by

$$\{dx, dy_1, \dots, dy_b, xdz_1, \dots, xdz_f\}.$$

We denote  $p$ -th order exterior power of  ${}^{ie}T^*M$  by  ${}^{ie}\Lambda^p M$ .

Let  $E$  denote any flat Hermitian vector bundle over the edge manifold  $M$ , defined by a unitary representation  $\rho : \pi_1(\overline{M}) \rightarrow U(\ell)$ . Such vector bundles are defined on all of  $\overline{M}$  and not only on the regular part of  $M$ .

If  $M$  is spin, we denote by  $S$  the corresponding spinor bundle. We define the spaces of edge differential operators  $\text{Diff}_e^k(M, {}^{ie}\Lambda^*T^*M \otimes E)$  and  $\text{Diff}_e^k(M, S \otimes E)$  as spaces of differential operators over  $M$  acting on compactly supported sections  $\Gamma_0({}^{ie}\Lambda^*T^*M \otimes E)$  and  $\Gamma_0(S \otimes E)$ , respectively, and given locally over the singular neighborhood by a sum of products of elements of edge vector fields  $\mathcal{V}_e$  with smooth (on  $M_c$ ) matrix-valued coefficients, with respect to suitable local trivializations of the vector bundles. Thus any  $L \in \text{Diff}_e^k$  is of the form

$$L = \sum_{j+|\alpha|+|\beta| \leq k} a_{j,\alpha,\beta}(x, y, z) (x\partial_x)^j (x\partial_y)^\alpha \partial_z^\beta,$$

with each  $a_{j,\alpha,\beta}$  matrix-valued and smooth up to  $x = 0$ .

A differential operator  $P$  of order  $k$  is an *incomplete* edge operator if  $x^k P \in \text{Diff}_e^k$ , i.e. if  $P \in x^{-k} \text{Diff}_e^k$ . Moreover, we denote by  $L^2$  the  $L^2$ -closure of compactly supported sections of the respective vector bundles and define the maximal domain of  $P$  by

$$(2.2) \quad \mathcal{D}_{\max}(P) := \{u \in L^2 \mid Pu \in L^2\},$$

where  $Pu \in L^2$  is understood in the distributional sense. The minimal domain  $\mathcal{D}_{\min}(D)$  is defined as a subspace of  $\mathcal{D}_{\max}(D)$ , given by the graph closure of  $\Gamma_0$ , where the graph norm is given in terms of the  $L^2$  norm  $\|\cdot\|_{L^2}$  by  $\|u\| := \|u\|_{L^2} + \|Pu\|_{L^2}$ .

We say that  $P$  is *locally independent of the twisting vector bundle  $E$*  up to conjugation if the following is true, cf. §3.3 for a more detailed account. Consider another flat vector bundle  $E'$  of the same rank, defined by a representation of the fundamental group  $\pi_1(\overline{M})$ . Fix after an eventual refinement a trivializing atlas for both  $E$  and  $E'$ , for which the transition functions are locally constant. Near the edge singularity we can take such a trivializing atlas made of distinguished neighbourhoods of the form  $\mathcal{V} \times \overline{\mathcal{C}(F)}$  where  $\mathcal{V} \subset B$  is a contractible open neighborhood. Each such neighborhood comes with a local bundle isomorphism

$$(2.3) \quad \theta : E \restriction \mathcal{V} \times \overline{\mathcal{C}(F)} \rightarrow E' \restriction \mathcal{V} \times \overline{\mathcal{C}(F)}.$$

We say that  $P$  is locally independent of the twisting flat vector bundle up to conjugation, if  $P \equiv P_E$  is associated by some geometric procedure to a flat vector bundle  $E$  over  $\overline{M}$  of fixed rank, such that for any two such vector bundles  $E$  and  $E'$

$$(2.4) \quad P_E \restriction \mathcal{V} \times \overline{\mathcal{C}(F)} = \theta^{-1} \circ P_{E'} \circ \theta \restriction \mathcal{V} \times \overline{\mathcal{C}(F)}.$$

A similar definition can be given for an operator  $P \in \text{Diff}_e^k$ . Clearly if  $P_E$  is of the form  $Q \otimes \text{Id}_E$  for some differential operator in  $\text{Diff}_e^k(M, {}^{ie}\Lambda^* T^* M)$  or  $\text{Diff}_e^k(M, S)$ , then  $P$  is independent of the twisting flat vector bundle.

Notice then that the canonical flat connections of  $E$  and  $E'$  restricted to  $M$  will have trivial connection one-forms in the fixed atlas (intersected with  $M$ ). Using the vanishing of the connection one-forms in the trivializing atlas we have chosen, we conclude that the three geometric Dirac operators, i.e. the signature, the Gauss Bonnet and the spin Dirac operators, associated to an *incomplete* edge metric and twisted by a flat unitary connection on the flat vector bundle  $E$ , are all examples of incomplete edge operators of order 1 which are locally independent of the twisting flat vector bundle up to conjugation.

**Definition 2.3.** We call a first order formally self-adjoint incomplete edge differential operator  $D \in x^{-1}\text{Diff}_e^1$  **allowable**, if  $D = D_{\text{geo}}^E + P$ , with  $D_{\text{geo}}^E$  one of the three geometric Dirac operators twisted by a flat vector bundle  $E$  and  $P \in \text{Diff}_e^1$  locally independent of the twisting flat vector bundle up to conjugation.<sup>3</sup>

We shall impose the following geometric Witt condition throughout the paper. First we treat the case in which  $D = D_{\text{geo}}$ . Consider the family of vertical Dirac operators  $D_F$  on the links  $(F, \kappa)$  associated to  $D_{\text{geo}}$ . Since  $F$  is a closed compact manifold,  $D_F$  is essentially self-adjoint and admits a discrete spectrum  $\text{Spec}(D_F)$  which is assumed to be independent of the base point.

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<sup>3</sup>We may in principle extend the class of possible perturbations  $P$  to include certain pseudo differential operators, subject to the condition that we can construct the heat kernel of  $D^2$  microlocally.

**Assumption 2.4.** If  $D$  is the spin Dirac operator then we assume that <sup>4</sup>

$$(2.5) \quad \text{Spec}(D_F) \cap \left(-\frac{1}{2}, \frac{1}{2}\right) = \emptyset.$$

If  $D$  is the Gauss-Bonnet operator or the signature operator then we require that

$$(2.6) \quad \text{Spec}(D_F) \cap (-1, 1) = \{0\}, \quad H^k(F) = 0 \text{ if } \dim F = 2k.$$

If we twist by a flat vector bundle  $E$  then we leave these two geometric Witt conditions unchanged, since by contractibility of the fibres  $\overline{\mathcal{C}(F)}$  in the fibre bundle  $\phi : \overline{\mathcal{U}} \rightarrow B$ , the vector bundle  $E$  is trivial over each link  $F$ . Finally, if  $D$  is allowable,  $D = D_{\text{geo}}^E + P$ , then we say that the geometric Witt condition is satisfied by  $D$  if it is satisfied by  $D_{\text{geo}}$ .

Using the heat kernel construction of the second named author in [VER15], one should be able to relax these conditions to the following ones:

$$(2.7) \quad 0 \notin \text{Spec}(D_F) \text{ for the spin Dirac operator};$$

$$(2.8) \quad H^k(F) = 0 \text{ if } \dim F = 2k \text{ for the Gauss-Bonnet and the signature operator.}$$

A comment about essential self-adjointness of  $D$ . In case of the spin Dirac operator acting on sections of the spinor bundle  $S$  essential self-adjointness of  $D$  as an unbounded operator in the space of square integrable sections  $L^2(M, S)$  with core domain  $\Gamma_c(M, S)$  is a consequence <sup>5</sup> of the geometric Witt condition (2.5), see Albin and Gell-Redman [ALGE13, Theorem 1]. Building on the work of Chou, Albin and Gell-Redman also prove that if the scalar curvature is non-negative and it is strictly positive somewhere, then the geometric Witt condition (2.5) is satisfied.

The geometric Witt condition (2.6) in the Hodge - de Rham setting also ensures essential self-adjointness of the Gauss-Bonnet and the signature operators. This is implicit in the work of Cheeger and addressed explicitly in the work of Albin, Leichtnam, Mazzeo and Piazza [ALMP12, Proposition 5.11].

By inspection of the proofs in [ALGE13, Theorem 1] and [ALMP12, Proposition 5.11] and using the triviality of a flat bundle in a distinguished neighbourhood of a point  $b \in B$  one can establish <sup>6</sup> the following

**Proposition 2.5.** *If  $D$  is allowable and satisfies the geometric Witt condition then  $\mathcal{D}_{\min}(D) = \mathcal{D}_{\max}(D)$  and in particular,  $D$  is essentially self-adjoint.*

The geometric Witt condition for the Gauss-Bonnet and for the signature operator is obviously stronger than the cohomological Witt condition  $H^{f/2}(F) = 0$ , in the sense that it also requires absence of *small* eigenvalues, compare (2.6) and (2.8) where the former rules out small eigenvalues while the latter allows them. Similarly, condition (2.5) is obviously stronger than condition (2.7) in that it requires also the vanishing of small eigenvalues. If these small eigenvalues are present, the relevant geometric Dirac

<sup>4</sup>For the spin Dirac operator our notion of geometric Witt condition comes from [ALGE13].

<sup>5</sup>In fact, in the setting of isolated conical singularities the geometric Witt condition is equivalent to essential self-adjointness of the spin Dirac operator, see for instance [CHO85, Theorem 3.2]

<sup>6</sup>The indicial and normal operator at  $y \in B$  do not see the higher order perturbation  $P$  and do not see the twisting flat bundle.

operators are not essentially self-adjoint; in this case one can impose algebraic boundary conditions as in [VER15] and still obtain self-adjoint extensions of our Dirac operators. However, the generalisation of the results of this paper requires an extension of the microlocal heat kernel construction to these self-adjoint extensions. We leave all this to future investigations.

**2.2. A remark on iterated cone-edge singularities.** A priori the links  $F$  of an edge manifold are set to be smooth, compact and without boundary. The underlying reason for that restriction is that our analytic arguments here require a microlocal description of the heat kernel on such an edge manifold. To the best knowledge of the authors such a microlocal heat kernel analysis is so far not available on stratified spaces of arbitrary depth.

However, the analytic arguments here extend to a particular case of stratified pseudo-manifolds: namely the bottom stratum is equal to a point and the Riemannian metric near that singularity is exact. Consequently, from here on, we include the following higher depth stratified spaces in our class of (admissible) edge manifolds: the bottom stratum is an isolated point, the Riemannian metric  $g \upharpoonright \mathcal{U} = dx^2 + x^2\kappa$  is an exact conical metric, and the regular part of  $F$  together with  $\kappa$  define a compact stratified space such that  $D_F$  is essentially self-adjoint and admits discrete spectrum. For example, the signature operator on a finite cone over an odd dimensional Witt space of depth  $\ell$  would satisfy this hypothesis.

### 3. MICROLOCAL ANALYSIS OF THE HEAT KERNEL ON EDGES

**3.1. Heat kernel asymptotics for the Hodge Laplacian.** In this subsection we consider the Friedrichs self adjoint extension of the Hodge Laplace operator on  $(M, g)$  twisted with a flat connection on a flat Hermitian vector bundle  $(E, \nabla, h)$ , which we denote by  $\Delta$  again, with domain  $\mathcal{D}(\Delta)$ . The corresponding heat operator acts as an integral convolution operator on  $u \in \mathcal{D}(\Delta)$

$$(3.1) \quad e^{-t\Delta}u(t, p) = \int_M (H_{\mathcal{F}}(t, p, \tilde{p}), u(\tilde{p}))_{g,h} \mathrm{dvol}_g(\tilde{p}),$$

where the heat kernel  $H_{\mathcal{F}}$  is a distribution on  $M_h^2 = \mathbb{R}^+ \times M_c^2$ , taking values in  $({}^{ie}\Lambda^*T^*M \otimes E) \boxtimes ({}^{ie}\Lambda^*T^*M \otimes E)$ . Its pointwise action on  $u$  is given by taking the pointwise inner product defined by the Riemannian edge metric  $g$  and the Hermitian bundle metric  $h$ . Then  $e^{-t\Delta}u$  solves the homogeneous heat problem

$$(\partial_t + \Delta)\omega(t, p) = 0, \quad \omega(0, p) = u(p),$$

for any  $u \in \mathcal{D}(\Delta)$ . We proceed with discussing the asymptotic properties of  $H_{\mathcal{F}}$ . Consider the local coordinates near the corner in  $M_h^2$  given by  $(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}))$ , where  $(x, y, z)$  and  $(\tilde{x}, \tilde{y}, \tilde{z})$  are two copies of coordinates on  $M$  near the edge. The kernel  $H_{\mathcal{F}}(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}))$  has non-uniform behaviour at the submanifolds

$$\begin{aligned} A &= \{(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) \in M_h^2 \mid t = 0, x = \tilde{x} = 0, y = \tilde{y}\}, \\ D &= \{(t, p, \tilde{p}) \in M_h^2 \mid t = 0, p = \tilde{p}\}, \end{aligned}$$

which requires an appropriate blowup of the heat space  $M_h^2$ , such that the corresponding heat kernel lifts to a polyhomogeneous distribution in the sense of the following definition, which we cite from [MEL93] and [MAVE12].

**Definition 3.1.** Let  $\mathfrak{W}$  be a manifold with corners and  $\{(H_i, \rho_i)\}_{i=1}^N$  an enumeration of its (embedded) boundary hypersurfaces with the corresponding defining functions. For any multi-index  $b = (b_1, \dots, b_N) \in \mathbb{C}^N$  we write  $\rho^b = \rho_1^{b_1} \dots \rho_N^{b_N}$ . Denote by  $\mathcal{V}_b(\mathfrak{W})$  the space of smooth vector fields on  $\mathfrak{W}$  which lie tangent to all boundary faces. A function  $\omega \in \rho^b L^\infty(\mathfrak{W})$  for some  $b \in \mathbb{C}^N$  is said to be conormal, if  $V_1 \dots V_\ell \omega \in \rho^b L^\infty(\mathfrak{W})$  for all  $V_j \in \mathcal{V}_b(\mathfrak{W})$  and for every  $\ell \geq 0$ . An index set  $E_i = \{(\gamma, p)\} \subset \mathbb{C} \times \mathbb{N}_0$  satisfies the following hypotheses:

- (i)  $\operatorname{Re}(\gamma)$  accumulates only at  $+\infty$ ,
- (ii) for each  $\gamma$  there exists  $P_\gamma \in \mathbb{N}_0$ , such that  $(\gamma, p) \in E_i$  for all  $p \leq P_\gamma$ ,
- (iii) if  $(\gamma, p) \in E_i$ , then  $(\gamma + j, p') \in E_i$  for all  $j \in \mathbb{N}_0$  and  $0 \leq p' \leq p$ .

An index family  $E = (E_1, \dots, E_N)$  is an  $N$ -tuple of index sets. Finally, we say that a conormal distribution  $\omega$  is polyhomogeneous on  $\mathfrak{W}$  with index family  $E$ , we write  $\omega \in \mathcal{A}_{\text{phg}}^E(\mathfrak{W})$ , if  $\omega$  is conormal and if in addition, near each  $H_i$ ,

$$\omega \sim \sum_{(\gamma, p) \in E_i} a_{\gamma, p} \rho_i^\gamma (\log \rho_i)^p, \text{ as } \rho_i \rightarrow 0,$$

where we require inductively that coefficients  $a_{\gamma, p}$  are conormal on  $H_i$ , i.e. polyhomogeneous with index  $E_j$  at any other boundary face  $H_j$  intersecting  $H_i$ .

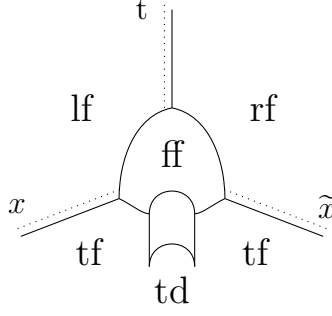
Blowing up the submanifolds  $A$  and  $D$  is a geometric procedure of introducing polar coordinates on  $M_h^2$ , around the submanifolds together with the minimal differential structure which turns polar coordinates into smooth functions on the blowup. A detailed account on the blowup procedure is given e.g. in [MEL93]. See also [GRI01].

First we blow up parabolically (i.e. we treat  $\sqrt{t}$  as a smooth variable) the submanifold  $A$ . This defines  $[M_h^2, A]$  as the disjoint union of  $M_h^2 \setminus A$  with the interior spherical normal bundle of  $A$  in  $M_h^2$ , equipped with the minimal differential structure such that smooth functions in the interior of  $M_h^2$  and polar coordinates on  $M_h^2$  around  $A$  are smooth. The interior spherical normal bundle of  $A$  defines a new boundary hypersurface – the front face  $\text{ff}$  in addition to the previous boundary faces  $\{x = 0\}$ ,  $\{\tilde{x} = 0\}$  and  $\{t = 0\}$ , which lift to  $\text{rf}$  (the right face),  $\text{lf}$  (the left face) and  $\text{tf}$  (the temporal face), respectively.

The actual heat-space  $\mathcal{M}_h^2$  is obtained by a second parabolic blowup of  $[M_h^2, A]$  along the diagonal  $D$ , lifted to a submanifold of  $[M_h^2, A]$ . We proceed as before by cutting out the lift of  $D$  and replacing it with its spherical normal bundle, which introduces a new boundary face – the temporal diagonal  $\text{td}$ . The heat space  $\mathcal{M}_h^2$  is illustrated in Figure 1.

We now describe projective coordinates in a neighborhood of the front face in  $\mathcal{M}_h^2$ , which are used often as a convenient replacement for the polar coordinates. The drawback is that projective coordinates are not globally defined over the entire front face.



FIGURE 1. The heat-space  $\mathcal{M}_h^2$ .

Near the top corner of the front face  $\text{ff}$ , projective coordinates are given by

$$(3.2) \quad \rho = \sqrt{t}, \quad \xi = \frac{x}{\rho}, \quad \tilde{\xi} = \frac{\tilde{x}}{\rho}, \quad u = \frac{y - \tilde{y}}{\rho}, \quad z, \quad \tilde{y}, \quad \tilde{z}.$$

With respect to these coordinates,  $\rho, \xi, \tilde{\xi}$  are in fact the defining functions of the boundary faces  $\text{ff}$ ,  $\text{rf}$  and  $\text{lf}$  respectively. For the bottom right corner of the front face, projective coordinates are given by

$$(3.3) \quad \tau = \frac{t}{\tilde{x}^2}, \quad s = \frac{x}{\tilde{x}}, \quad u = \frac{y - \tilde{y}}{\tilde{x}}, \quad z, \quad \tilde{x}, \quad \tilde{y}, \quad \tilde{z},$$

where in these coordinates  $\tau, s, \tilde{x}$  are the defining functions of  $\text{tf}$ ,  $\text{rf}$  and  $\text{ff}$  respectively. For the bottom left corner of the front face, projective coordinates are obtained by interchanging the roles of  $x$  and  $\tilde{x}$ . Projective coordinates on  $\mathcal{M}_h^2$  near the temporal diagonal are given by

$$(3.4) \quad \eta = \frac{\sqrt{t}}{\tilde{x}}, \quad S = \frac{(x - \tilde{x})}{\sqrt{t}}, \quad U = \frac{y - \tilde{y}}{\sqrt{t}}, \quad Z = \frac{\tilde{x}(z - \tilde{z})}{\sqrt{t}}, \quad \tilde{x}, \quad \tilde{y}, \quad \tilde{z}.$$

In these coordinates,  $\text{tf}$  is defined as the limit  $|(S, U, Z)| \rightarrow \infty$ , where we use the Euclidean norm on  $\mathbb{R}^m$ ;  $\text{ff}$  and  $\text{td}$  are defined by  $\tilde{x}, \eta$ , respectively. The blow-down map  $\beta : \mathcal{M}_h^2 \rightarrow M_c^2$  is in local coordinates simply the coordinate change back to  $(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}))$ .

The restriction of  $\beta$  to  $\text{td}$  is a fibration, with each fibre being a closed ‘parabolic’ hemisphere  $S_+^m$  and base given by the lifted diagonal of  $(M_c)^2$ , which is diffeomorphic to a copy of  $M_c$ . Similarly, the restriction of  $\beta$  to  $\text{ff}$  is a fibration over the diagonal of  $\partial M_c \times \partial M_c$  with fibres being the ‘parabolic’ quarter-spheres  $S_{++}^{m+1}$ .

The heat kernel is constructed by the second named author jointly with Mazzeo in [MAVE12] as a polyhomogeneous conormal distribution on  $\mathcal{M}_h^2$  with product type expansions at the corners of the heat space, and with coefficients depending smoothly on the tangential variables. [MAVE12] asserts the following result.

**Theorem 3.2.** *Let  $(M, g)$  be an incomplete Riemannian manifold with an edge singularity and an admissible edge metric  $g$ . Consider a flat Hermitian vector bundle over  $M$ . Then the lift  $\beta^* H_{\mathcal{F}}$  of the Friedrichs heat kernel is a polyhomogeneous function on*

$\mathcal{M}_h^2$  taking values in  $({}^{ie}\Lambda^*T^*M \otimes E) \boxtimes ({}^{ie}\Lambda^*T^*M \otimes E)$  with the index set  $(-m + \mathbb{N}_0, 0)$  at  $\text{ff}$ ,  $(-m + \mathbb{N}_0, 0)$  at  $\text{td}$ , vanishing to infinite order at  $\text{tf}$ , and with some discrete index set at  $\text{rf}$  and  $\text{lf}$ .

To be precise, [MAVE12] establishes the heat kernel asymptotics under certain rescaling transformation, as employed by Brüning and Seeley in [BRSE88]. In view of the transformation rules in [MAVE12, (3.13)] it is straightforward to derive the index set of the heat kernel at  $\text{ff}$  without using the rescaling.

We point out that the heat kernel construction extends to squares of allowable Gauss-Bonnet or signature operators, in the sense of Definition 2.3.

**3.2. Heat kernel asymptotics for the spin Laplacian.** Assume  $M$  is spin and consider the spin bundle  $S$  on  $M$ . We denote the corresponding spin Dirac operator by  $D$ . Recall that  $D$  is essentially self-adjoint under the geometric Witt assumption 2.4. Assume for the moment that the higher order term  $h$  in the Riemannian metric  $g$  on  $M$  is zero. Then the explicit structure of  $D$  is studied in [ALGE13].

Since by the admissibility assumption,  $\phi : (\partial M, \kappa + \phi^*g^B) \rightarrow (B, g^B)$  is a Riemannian submersion, the tangent bundle  $T_p\partial M$  splits into vertical and horizontal subspaces  $T_p^V\partial M \oplus T_p^H\partial M$ , where the vertical subspace  $T_p^V\partial M$  is the tangent space to the fibre of  $\phi$  through  $p$ , the horizontal subspace  $T_p^H\partial M = \phi^*TB$  is the pull back of the space of tangent vectors of the base, lifted to  $Y$  and extended trivially inwards, and the tensor  $\kappa$  restricted to  $T_p^H\partial M$  vanishes. The splitting of the tangent bundle over  $\partial M$  yields a splitting of the tangent bundle of  $M$  near the edge

$$(3.5) \quad {}^{ie}TM = \langle \partial_x \rangle \oplus x^{-1}T^V\partial M \oplus \phi^*TB.$$

Since  $\kappa$  restricted to  $\phi^*TB$  vanishes, there exists a local orthonormal frame of vector fields  $\{\partial_x, x^{-1}V_\alpha, U_\beta\}$ , which respects the splitting (3.5). Denote by  $\nabla$  the covariant derivative on the spin bundle, induced from the lift of the Levi-Civita connection to the spin principle bundle. Then, by [ALGE13, Lemma 1.1] the Dirac operator  $D$  decomposes locally as

$$(3.6) \quad D = c(\partial_x)\partial_x + \frac{f}{2x}c(\partial_x) + \frac{1}{x} \sum_{\alpha=1}^f c(x^{-1}V_\alpha)\nabla_{V_\alpha} + \sum_{\beta=1}^b c(U_\beta)\nabla_{U_\beta} + W,$$

where  $c$  denotes the Clifford multiplication and  $W$  a higher order term. Allowing for non-zero higher term  $h$  in the Riemannian metric  $g$  just introduces higher order terms in the formula for  $D$ . From here it is clear that the normal operator  $N(x^2D)_{y_0}$  is  $s^2$  times the spin Dirac operator on the model edge  $\mathbb{R}_s^+ \times F \times \mathbb{R}^b$  with the model edge metric  $g_{ie} = ds^2 + s^2\kappa_{y_0} + g_{y_0}^B$ , in a similar way as in the Hodge de Rham setting.

We can now construct the heat kernel  $H$  of the spin Laplacian  $D^2$  as a polyhomogeneous function on the blowup space  $\mathcal{M}_h^2$  taking values in  $S \boxtimes S$  along the lines of [MAVE12]. The normal operator  $N(H)_{y_0}$ , which is the leading order term of the heat kernel at the  $y_0$ -fibre of the front face, is obtained as a product of the heat kernel  $H_{\mathcal{C}(F)}$  on the model cone  $(\mathbb{R}_s^+ \times F, ds^2 + s^2\kappa_{y_0})$  with the heat kernel on the euclidean factor  $(\mathbb{R}^b, g_{y_0}^B)$ .

The spin Laplacian  $\Delta_{\mathcal{C}(F)}$  and the spin heat kernel  $H_{\mathcal{C}(F)}$  on the model cone has been identified explicitly by Chou [CHO85, (2.6), (2.8)]. More precisely, [CHO85, (2.19)] asserts that any eigenspinor of  $D$  is of the form  $\omega_\mu(\phi_\mu \pm c(\partial_s)\phi_\mu)$  with the scalar part  $\omega_\mu$  given in terms of Bessel functions of first kind, and  $\phi_\mu$  being a  $\mu$ -eigenspinor of the Dirac operator  $D_F = \sum c(V_\alpha)\nabla_{V_\alpha}$  on the fibres  $(F, \kappa)$ . We write  $S_\mu^\pm := (\phi_\mu \pm c(\partial_s)\phi_\mu)$ . The action of the spin Laplacian on spinors of the form  $\omega S_\mu^\pm$  with  $\omega \in C_0^\infty(\mathbb{R}^+)$  amounts then by [CHO85, Prop. 2.5, (2.6), (2.7)] to a scalar multiplication operator

$$(3.7) \quad \Delta_{\mathcal{C}(F)}(\omega S_\mu^\pm) = \left( -\partial_s^2 - \frac{f}{s}\partial_s + \frac{1}{s^2} \left( \mu^2 \mp \mu - \frac{f^2 - 2f}{4} \right) \omega \right) S_\mu^\pm.$$

The heat kernel of this scalar action is well-known and expressed in terms of modified Bessel functions in [LES97, Proposition 2.3.9]. Correspondingly, the spin heat kernel  $H_{\mathcal{C}(F)}$  is written out in [CHO85, (5.9)] with  $\nu^\pm(\mu) := |2\mu \mp 1|/2$  as

$$(3.8) \quad H_{\mathcal{C}(F)} = \sum_\mu H_\mu^\pm S_\mu^\pm \otimes S_\mu^\pm, \quad H_\mu^\pm(t, s, \tilde{s}) = \frac{1}{2t} (s\tilde{s})^{(1-f)/2} I_{\nu^\pm(\mu)} \left( \frac{s\tilde{s}}{2t} \right) e^{-\frac{s^2 + \tilde{s}^2}{4t}}.$$

Hence the normal operator  $N(H)_{y_0}$  can be set up exactly as in [MAVE12, (3.10)]. In order to construct the exact heat kernel, the initial parametrix  $N(H)_{y_0}$  has to be corrected, which involves composition of Schwartz kernels on  $\mathcal{M}_h^2$  by [MAVE12, Theorem 5.3]. Following the heat kernel construction as in [MAVE12] verbatim we arrive at the following result.

**Theorem 3.3.** *Let  $(M, g)$  be an incomplete Riemannian spin manifold with an edge singularity and an admissible edge metric  $g$ . Assume the geometric Witt condition. Then the spin Dirac operator  $D$  is essentially self-adjoint and the lift  $\beta^*H$  of the heat kernel for the spin Laplacian  $D^2$  is a polyhomogeneous function on  $\mathcal{M}_h^2$  taking values in  $S \boxtimes S$  with the index set  $(-m + \mathbb{N}_0, 0)$  at  $ff$ ,  $(-m + \mathbb{N}_0, 0)$  at  $td$ , vanishing to infinite order at  $tf$ , and with some discrete index set at  $rf$  and  $lf$ .*

We point out that the heat kernel construction extends to squares of allowable Dirac operators satisfying the Witt condition in the sense of Definition 2.3.

**3.3. On locality of heat kernel asymptotics along the edge.** The heat kernel construction for the Hodge and spin Laplacians and their higher order (allowable) perturbations carries over verbatim to the case where the operators are twisted with a flat vector bundle  $E$ . We consider flat vector bundles induced by unitary representations  $\rho : \pi_1(\overline{M}) \rightarrow U(n)$ , where  $\overline{M}$  is the smoothly stratified space; notice that, therefore, the flat vector bundles here are defined on all of  $\overline{M}$  and not only on the regular part  $M$ .

Let  $\overline{\mathcal{U}}$  be a tubular neighbourhood of the singular stratum  $B$ ; from the axioms of Thom-Mather we know that  $\overline{\mathcal{U}}$  is a fiber bundle over  $B$ ,  $\phi : \overline{\mathcal{U}} \rightarrow B$ , with fiber equal to  $\mathcal{C}(F)$ . As already remarked, since the fibres  $\mathcal{C}(F)$  are contractible,  $E$  is trivial when restricted to a distinguished neighbourhood  $\mathcal{V} \times \mathcal{C}(F)$  of a point  $y_0 \in \mathcal{V} \subset B$ , with  $\mathcal{V}$  a contractible open neighbourhood of  $y_0$  in  $B$ . In particular,  $E$  is trivial over each link  $F$ , but not necessarily along the edge singularity  $B$ .

We seek to clarify here to what extent the asymptotic behaviour of the heat kernel  $H$  near the edge depends on the twisting bundle  $E$ . To this end, we consider two such flat vector bundles  $E_1$  and  $E_2$  of rank  $r$  and fix after an eventual refinement a trivializing atlas for both  $E_1$  and  $E_2$ , for which the transition functions are locally constant. We can take such a trivializing atlas made of distinguished neighbourhoods of the form  $\mathcal{V} \times \overline{\mathcal{C}(F)}$  near  $B$ . Each such neighborhood comes with a local bundle isomorphism

$$(3.9) \quad \phi : E_1 \upharpoonright \mathcal{V} \times \overline{\mathcal{C}(F)} \rightarrow E_2 \upharpoonright \mathcal{V} \times \overline{\mathcal{C}(F)}.$$

Notice then that the restriction of each  $E_j, j = 1, 2$ , to  $M$  will have a flat connection for which the connection one-forms associated to this atlas (intersected with  $M$ ) are zero. We fix this flat connection once and for all.

Consider the front face of the heat space  $\mathcal{M}_h^2$ , which is a fibration over  $\partial M$  and as such can also be viewed as a fibration over  $B$ . Fix a base point  $y_0 \in B$ . Intersection of the distinguished neighborhood  $\mathcal{V} \times \overline{\mathcal{C}(F)}$  of  $y_0$  with the open smooth stratum  $M$  is given by  $\mathcal{V} \times \mathcal{C}(F)$ . We denote the Hodge or the spin Laplacian twisted with  $E_j$  by  $\Delta_j$  for  $j = 1, 2$ . We denote their heat kernels by the corresponding lower index  $j$  as well. Using the vanishing of the connection one forms of  $E_j$  in the trivializing atlas we have chosen, we conclude that over  $\mathcal{V} \times \mathcal{C}(F)$  the operators are related by

$$(3.10) \quad \Delta_1 = \phi^{-1} \circ \Delta_2 \circ \phi.$$

The lifts of  $\Delta_1$  and  $\Delta_2$  to  $\mathcal{M}_h^2$  are again related via conjugation by the local bundle isomorphism  $\phi$  over  $\mathcal{V} \times \mathcal{C}(F)$ . Hence, for  $\tilde{y} \in \mathcal{V}$  the normal operators  $N(H_1)_{\tilde{y}}$  and  $N(H_2)_{\tilde{y}}$ , which are set up exactly as in [MAVE12, (3.10)], are related via conjugation by  $\phi$  as well. The initial parametrices  $H_0(\Delta_j)$  are obtained by extending  $N(H_j)_{\tilde{y}}$  smoothly off the front face to the interior of  $\mathcal{M}_h^2$  for any  $j = 1, 2$ . By construction, the front face asymptotics of  $H_0(\Delta_j)(\cdot, \tilde{y})$  with  $\tilde{y} \in \mathcal{V}$  is, up to conjugation, independent of  $j$ , i.e. writing  $\rho_{\text{ff}}$  for the defining function of the front face in  $\mathcal{M}_h^2$  we have for any  $N \in \mathbb{N}$

$$(3.11) \quad (H_0(\Delta_1) - \phi^{-1} \circ H_0(\Delta_2) \circ \phi)(\cdot, \tilde{y}) = O(\rho_{\text{ff}}^N), \quad \rho_{\text{ff}} \rightarrow 0.$$

The initial parametrix  $H_0(\Delta_j)$  solves the heat equation only up to an error  $R(\Delta_j) := (\partial_t + \Delta_j)H_0(\Delta_j)$ . For  $\tilde{y} \in \mathcal{V}$  the front face asymptotics of  $R(\Delta_j)(\cdot, \tilde{y})$  is again as in (3.11) independent of  $j$  up to conjugation by  $\phi$ , since the same holds for  $H_0(\cdot, \tilde{y})$  and  $\Delta$  in the open neighborhood  $\mathcal{V} \times \mathcal{C}(F)$ . Now the parametrix  $H_0(\Delta_j)$  is corrected to define an exact solution to the heat equation by adding convolutions  $H_0(\Delta_j) * R(\Delta_j)^\ell$  for  $\ell \in \mathbb{N}$ . These convolutions involve integrations over the base  $B$  and hence a priori might depend on the global structure of  $E_1$  and  $E_2$ .

In order to address this issue, consider a kernel  $P$  that lifts to a polyhomogeneous function  $\beta^*P$  on  $\mathcal{M}_h^2$ . Consider any  $y \in B$  with  $|y - \tilde{y}| \geq \delta$  bounded away from zero. We write  $u = (y - \tilde{y})/\rho_{\text{ff}}$  in correspondence with the projective coordinates in (3.2) and (3.3). Assume that  $P$  admits a factor of the form  $\exp(-\frac{1}{2}|u|^2)$ . Then for any  $N \in \mathbb{N}$  we may estimate

$$(3.12) \quad \begin{aligned} \rho_{\text{ff}}^{-N} \|P(\cdot, y, \tilde{y}, \rho_{\text{ff}})\|_\infty &= |u|^N \|P(\cdot, y, \tilde{y}, \rho_{\text{ff}})\|_\infty |y - \tilde{y}|^{-N} \\ &\leq |u|^N \|P(\cdot, y, \tilde{y}, \rho_{\text{ff}})\|_\infty \delta^{-N}. \end{aligned}$$

and the right hand side is bounded by a constant  $C_N$  thanks to the factor  $\exp(-\frac{1}{2}|u|^2)$  in  $P$ . Consequently,  $\beta^*P$  is vanishing to infinite order at the front face, when evaluated at two points  $y$  and  $\tilde{y}$  on the edge  $B$  with their distance bounded away from zero.

Consider now the convolutions  $H_0(\Delta_j) * R(\Delta_j)^\ell$ , whose definition involves integrations of the kernels over the base  $B$ . Since the initial parametrix  $H_0(\Delta_j)$  admits a factor of the form  $\exp(-\frac{1}{2}|u|^2)$  by definition, this factor is present in the error term  $R(\Delta_j)$  and the related convolutions  $H_0(\Delta_j) * R(\Delta_j)^\ell$  as well. Then (3.12) shows that the contributions to  $H_0(\Delta_j) * R(\Delta_j)^\ell(\cdot, \tilde{y})$  coming from integration on  $B$  away from a  $\tilde{y}$  are vanishing to infinite order at ff. Consequently, the front face asymptotics of  $H_0(\Delta_j) * R(\Delta_j)^\ell(\cdot, \tilde{y})$  depends only on the asymptotics of  $H_0(\Delta_j)(\cdot, y)$  and  $R(\Delta_j)(\cdot, y)$  for  $y$  sufficiently close to  $\tilde{y} \in \mathcal{V}$ , and hence

$$(3.13) \quad (H(\Delta_1) - \phi^{-1} \circ H(\Delta_2) \circ \phi)(\cdot, \tilde{y}) = O(\rho_{\text{ff}}^N), \quad \rho_{\text{ff}} \rightarrow 0.$$

This proves that the front face asymptotics of the lifted heat kernel  $\beta^*H(\cdot, \tilde{y})$  is up to conjugation of the coefficients, independent of the choice of a flat vector bundle of fixed rank, for  $\tilde{y}$  in an open neighborhood of  $y_0$ . A similar argument applies for the expansion of the heat kernel near the temporal diagonal. If we restrict the heat kernel to the diagonal and take its pointwise trace, then dependence of the expansions on the conjugation by a bundle morphism cancels and we arrive at the following theorem.

**Theorem 3.4.** *Let  $E$  be a flat vector bundle induced by a unitary representation of the fundamental group of  $\overline{M}$ . Consider the heat kernel  $H$  of the square of an allowable Dirac operator  $D$ , twisted by the flat vector bundle  $E$ . The asymptotic expansion of the pointwise traces  $\text{tr } H$  and  $\text{tr } DH$  at ff and td is up to the bundle rank independent of the particular choice of  $E$ .*

#### 4. THE EVEN AND ODD HEAT CALCULUS ON EDGE SPACES

Admissibility of  $g$ , more precisely evenness of the higher order term  $h$  in its behavior near the edge, in fact yields a more refined asymptotic information beyond Theorem 3.2. This refinement is obtained by singling out even and odd polyhomogeneous functions on the blowup space  $\mathcal{M}_h^2$ . The even/odd classification is slightly different for the Hodge Laplacian and for the spin Laplace operator and hence done here separately. As before, we assume in both cases the geometric Witt condition, so that the corresponding Dirac operators are essentially self-adjoint and the heat kernels for their squares can be studied as polyhomogeneous functions on the heat space  $\mathcal{M}_h^2$ .

**4.1. The even and odd heat calculus for the Hodge Laplacian.** We begin with the explicit definition of a heat calculus.

**Definition 4.1.** Let  $\mathcal{E} = (E_{\text{lf}}, E_{\text{rf}})$  be an index family for the two side faces of  $\mathcal{M}_h^2$ . We define  $\Psi_{\text{e-h}}^{\ell, p, \mathcal{E}}(M)$  to be the space of all operators  $P$  with Schwartz kernels  $K_P$  which are pushforwards from polyhomogeneous functions  $\beta^*K_P$  on  $\mathcal{M}_h^2$ , taking values in  $({}^{ie}\Lambda^*T^*M \otimes E) \boxtimes ({}^{ie}\Lambda^*T^*M \otimes E)$  with index family  $(-m - 2 + \ell + \mathbb{N}_0, 0)$  at ff,  $(-m + p + \mathbb{N}_0, 0)$  at td,  $\emptyset$  at tf and  $\mathcal{E}$  for the two side faces of  $\mathcal{M}_h^2$ . The subscript e - h in  $\Psi_{\text{e-h}}^{\ell, p, \mathcal{E}}(M)$  stands for "edge-heat".

The naming *calculus* is due to composition formulae, which hold under certain restrictions on the index sets at the side faces

$$\Psi_{e-h}^{\ell,p,\mathcal{E}}(M) \circ \Psi_{e-h}^{\ell',p',\mathcal{E}'}(M) \subset \Psi_{e-h}^{\ell+\ell',p+p',\mathcal{E}''}(M),$$

where the index set  $\mathcal{E}''$  is defined in terms of the index sets  $(\mathcal{E}, \mathcal{E}')$  as in [MAVE12, Theorem 5.3].

We now introduce the definition of even and odd subcalculi, cf. [MAVE12]. Consider local coordinates  $(x, y, z)$  in  $\mathcal{U}$  and the local differential forms  $\{dx, xdz_1, \dots, xdz_f, dy_1, \dots, dy_b\}$  which generate the exterior algebra  ${}^{ie}\Lambda^*T^*M$  near the edge. We assign odd parity to the forms  $\{dx, xdz_1, \dots, xdz_f\}$  and even parity to the forms  $\{dy_1, \dots, dy_b\}$ . Exterior product of two even differential forms is set to be even again. Exterior product of two odd differential forms is set to be even as well. Exterior product of an odd and an even differential form is set to be odd. This decomposes the exterior algebra  ${}^{ie}\Lambda^*T^*M$  near the edge into an even  $\Lambda_+$  and an odd subbundle  $\Lambda_-$

$$(4.1) \quad {}^{ie}\Lambda^*T^*M \upharpoonright \mathcal{U} = \Lambda_+ \oplus \Lambda_-.$$

Evenness and oddness of differential forms, is well-defined within a fixed choice of special coordinates, as introduced in §2. Similarly, evenness of the admissible Riemannian metric  $g$  also depends on the choice of special coordinates, which we fix henceforth.

The assignment of the even/odd parity is not ad hoc and in no way arbitrary. Recall that the evenness condition on the Riemannian edge metric asserts that the coefficients of  $g$  are even functions of  $x$ , except for the coefficients corresponding to the  $dx$ -cross terms which are required to be odd in  $x$ . One may check explicitly that the same continues to hold for the inverse Riemannian metric on differential forms. Consequently,  $g(dx, dx), g(dy_i, dy_j), g(xdz_k, xdz_\ell)$  as well as  $g(dx, xdz_k)$  are always even functions of  $x$ , i.e. are smooth in  $M_c$  with a Taylor expansion at  $x = 0$  involving only powers of  $x^2$ . Moreover,  $g(dx, dy_j)$  as well as  $g(dy_i, xdz_j)$  are always odd functions of  $x$ , i.e. their Taylor expansions at  $x = 0$  involve only odd powers of  $x$ .

We denote the space of even functions on  $M$  by  $C_+^\infty(M)$ . We denote the space of odd functions on  $M$  by  $C_-^\infty(M)$ . Clearly,  $C_-^\infty(M) = xC_+^\infty(M)$ . Extending these relations to pairing of exterior products of cotangent vectors, we find for the pointwise traces

$$(4.2) \quad \begin{aligned} \text{tr}(\Lambda_\pm \boxtimes \Lambda_\pm) &= g(\Lambda_\pm, \Lambda_\pm) \subset C_+^\infty(M), \\ \text{tr}(\Lambda_\pm \boxtimes \Lambda_\mp) &= g(\Lambda_\pm, \Lambda_\mp) \subset C_-^\infty(M). \end{aligned}$$

where the equalities in (4.2) are justified by identifying  $\Lambda_\pm \boxtimes \Lambda_\pm$  and  $\Lambda_\pm \boxtimes \Lambda_\pm^*$  through the metric and using that  $\Lambda_\pm \boxtimes \Lambda_\pm^*|_\Delta = \text{Hom}(\Lambda_\pm, \Lambda_\pm)$ . We can now define the even/odd calculus by imposing conditions on the asymptotic expansions of elements in  $\Psi_{e-h}^*$  at the temporal diagonal  $\text{td}$  and the front face  $\text{ff}$ .

**Definition 4.2.** Consider  $P \in \Psi_{e-h}^*(M)$  with the Schwartz kernel  $K_P$ . Then  $P$  is said to be even if the following two conditions hold. At  $\text{td}$  we require in terms of the



projective coordinates (3.4), valid in the interior of td

$$\beta^* K_P \sim \eta^{-m-2} \sum_{n=0}^{\infty} \kappa_n^{\text{td}}(S, U, Z, \tilde{x}, \tilde{y}, \tilde{z}) \eta^n,$$

with  $\kappa_n^{\text{td}}(-S, -U, -Z, \tilde{x}, \tilde{y}, \tilde{z}) = (-1)^n \kappa_n^{\text{td}}(S, U, Z, \tilde{x}, \tilde{y}, \tilde{z})$  for all  $n$ .

At ff we require in terms of the projective coordinates (3.2), valid in the interior of ff

$$\beta^* K_P \sim \rho^{-m-2} \sum_{n=0}^{\infty} (\kappa_{n,\pm,\pm}^{\text{ff}} + \kappa_{n,\pm,\mp}^{\text{ff}})(\xi, \tilde{\xi}, u, z, \tilde{y}, \tilde{z}) \rho^n;$$

with  $\kappa_{n,\pm,\pm}^{\text{ff}}(\xi, \tilde{\xi}, -u, z, \tilde{y}, \tilde{z}) = (-1)^n \kappa_{n,\pm,\pm}^{\text{ff}}(\xi, \tilde{\xi}, u, z, \tilde{y}, \tilde{z})$  for all  $n$ ,  
with  $\kappa_{n,\pm,\mp}^{\text{ff}}(\xi, \tilde{\xi}, -u, z, \tilde{y}, \tilde{z}) = (-1)^{n+1} \kappa_{n,\pm,\mp}^{\text{ff}}(\xi, \tilde{\xi}, u, z, \tilde{y}, \tilde{z})$  for all  $n$ .

Here the coefficients  $\kappa_{n,\pm,\pm}^{\text{ff}}$  are set to take values in  $\Lambda_{\pm} \boxtimes \Lambda_{\pm}$ , while the coefficients  $\kappa_{n,\pm,\mp}^{\text{ff}}$  take values in  $\Lambda_{\pm} \boxtimes \Lambda_{\mp}$ .

We denote the set of all even operators by  $\Psi_+^*(M)$ . We define the odd subcalculus  $\Psi_-^*(M)$  by requiring the coefficients  $\kappa_{n,\pm,\pm}^{\text{ff}}$  to be odd and the coefficients  $\kappa_{n,\pm,\mp}^{\text{ff}}$  to be even. In other words,  $\Psi_-^*(M)$  is defined by replacing  $(-1)^n$  in the parity conditions at ff by  $(-1)^{n+1}$ . The parity at td remains unchanged.

Note that the parity of coefficients in the definition of the even subcalculus is defined precisely so that after restricting to the diagonal (the diagonal in  $\mathcal{M}_h^2$  is again a manifold with corners, introduced in detail in §5 with  $\eta$  still being the defining function of td, and  $\rho$  the defining function of ff) and taking pointwise trace, one finds in view of (4.2) for any  $P \in \Psi_+^*(M)$

$$\begin{aligned} \text{tr } \beta^* K_P &\sim \eta^{-m-2} \sum_{n=0}^{\infty} \kappa_{2n}^{\text{td}}(\tilde{x}, \tilde{y}, \tilde{z}) \eta^{2n}, \quad \eta \rightarrow 0, \\ \text{tr } \beta^* K_P &\sim \rho^{-m-2} \sum_{n=0}^{\infty} \kappa_{2n}^{\text{ff}}(\tilde{\xi}, \tilde{y}, \tilde{z}) \rho^{2n}, \quad \rho \rightarrow 0. \end{aligned} \tag{4.3}$$

Similar expansions hold of course for  $P \in \Psi_-^*(M)$  with the index  $2n$  being replaced by  $(2n+1)$ . We should point out that the even odd classification can equivalently be introduced by studying the sign change under the map  $(x, u) \mapsto (-x, -u)$ , cf. [ALB07], once evenness of the pointwise traces is taken into account. We now study the action of the geometric operators on  $(M, g)$  with respect to the even odd calculus.

**Proposition 4.3.** *The exterior derivative  $d$  and the Hodge star operator  $*$  acts as follows with respect to the even odd subcalculi*

$$d \circ \Psi_{\pm}^* \subset \Psi_{\pm}^*, \quad * \circ \Psi_{\pm}^* \subset \Psi_{\pm}^*, \text{ if } (m-b) \text{ is even; } * \circ \Psi_{\pm}^* \subset \Psi_{\mp}^*, \text{ if } (m-b) \text{ is odd.}$$

*In particular*

- (i) *the Hodge Laplacian respects the even subcalculus  $\Psi_+^*$ ,*
- (ii) *the Gauss Bonnet operator respects the even subcalculus  $\Psi_+^*$ ,*

- (iii) the signature operator<sup>7</sup> respects the even subcalculus  $\Psi_+^*$  only if  $b$  is even<sup>8</sup>.
- (iv) the odd signature operator respects the even subcalculus  $\Psi_+^*$  if  $(m - b)$  is even, and acts as  $D \circ \Psi_\pm^* \subset \Psi_\mp^*$  if  $(m - b)$  is odd.

*Proof.* Consider the individual components of the exterior derivative  $d$ . Its action leaves the parity of coefficients in the td asymptotics invariant. Indeed for instance  $\beta^*(\partial_x) = (\tilde{x}\eta)^{-1}\partial_S$ , where the lowering of td asymptotics by  $\eta^{-1}$  is offset by the parity change for the coefficients  $\partial_S \kappa_n^{\text{td}}$ .

At the front face, differentiation in  $x$  with  $\partial_x(\cdot) \wedge dx$  does not change the parity. Indeed, in (3.4) coordinates  $\beta^*(\partial_x) = \rho^{-1}\partial_\xi$  and hence differentiation in  $x$  lowers the front face behaviour by one and is an odd action. However  $dx$  is odd as well, so that the total action of  $\partial_x(\cdot) \wedge dx$  is even. The  $z$ -derivative component in  $d$  comes as  $x^{-1}\partial_z(\cdot) \wedge xdz$ . The factor  $x^{-1}$  decreases front face behaviour by one and hence is odd. However,  $xdz$  is odd as well, so that the total action of  $x^{-1}\partial_z(\cdot) \wedge xdz$  is even. Finally, the action of  $\partial_y(\cdot) \wedge dy$  is even also. Indeed,  $dy$  is even and  $\beta^*(\partial_y) = \rho^{-1}\partial_u$  respects the even odd parity as well.

Let us now consider the Hodge star operator  $*$  of  $(M, g)$ . Here we just need to count the number of odd components  $\{dx, xdz\}$  and the even components  $\{dy\}$ . We find that  $*$  :  $\Lambda_\pm \rightarrow \Lambda_\pm$  if  $(m - b)$  is even, and  $*$  :  $\Lambda_\pm \rightarrow \Lambda_\mp$  if  $(m - b)$  is odd. From there it is clear that  $*$  respects the even odd subcalculus if  $(m - b)$  is even, while it changes the parity if  $(m - b)$  is odd.

It remains to check the actions of the various geometric Dirac and Laplace operators. Action of the Hodge Laplacian, the Gauss-Bonnet and the odd signature operator follows directly from the properties of  $d$  and  $*$ . The statement on the signature operator (in case  $m = \dim M$  is even) is obtained as follows. Recall that the signature operator is the restriction of the Gauss-Bonnet operator to the  $(+1)$ -eigenspace of an involution  $\tau$  with  $\tau \upharpoonright \Lambda^p T^*M = i^{p(p-1)+m/2}*$ . Since for  $m$  and  $b$  even the Hodge star operator  $*$  respects the even and odd subcalculi, so does  $\tau$ . Hence the  $(+1)$ -eigenspace of  $\tau$  decomposes into a direct sum of  $\Lambda_+$  and  $\Lambda_-$  components. The arguments for the Gauss-Bonnet operator then carry over to the signature operator.

This argument fails for the signature operator in case  $b$  is odd. The  $(\pm 1)$ -eigenspace of  $\tau$  is spanned by elements of the form  $(\omega \pm \tau\omega)$  where  $\omega$  is any differential form on  $M$ . Since for  $b$  odd,  $\tau$  interchanges the even and odd differential forms, its  $(\pm 1)$  eigenspaces are comprised of elements of mixed parity and do not admit a direct sum decomposition into even and odd forms. In this case the signature operator acting on the  $(+1)$ -eigenspace of  $\tau$  does not any longer admit an even or odd classification.  $\square$

Analysis of the even odd parity for the exterior derivative extends to the twisted setting. As before, let  $E$  be a flat vector bundle on  $M$  corresponding to a unitary representation of the fundamental group of the singular space  $\overline{M}$ . This flat vector

<sup>7</sup>on an even-dimensional edge manifold

<sup>8</sup>when  $\dim F$  is odd and  $M$  is therefore Witt. The case of  $\dim F$  even is excluded in this statement.

bundle induces in a natural way a flat vector bundle on the resolved manifold  $M_c$ . Let  $Y = \partial M_c$ . We can and we shall assume that

$$(4.4) \quad E|_{\mathcal{U}} \cong \pi^*(E_Y) \cong (0, 1] \times E_Y,$$

where  $E_Y$  is the restriction of  $E$  to  $Y$ , and where  $\pi: (0, 1] \times Y \simeq \mathcal{U} \rightarrow Y$  is the canonical projection. We can endow the flat vector bundle with an hermitian metric and a covariant derivative which are compatible with (4.4). In particular we shall assume that the flat covariant derivative  $\nabla$  has a product structure near the boundary, i.e. there exists a  $\nabla_Y$  such that for  $s \in C^\infty((0, 1], C^\infty(E_Y)) \cong \Gamma(E|_{\mathcal{U}})$  we have

$$(4.5) \quad \nabla s = \frac{\partial s}{\partial x} \otimes dx + \nabla_Y s.$$

Extend the covariant derivative to a linear operator on twisted differential forms  $\nabla: \Omega^k(M, E) \rightarrow \Omega^{k+1}(M, E)$ . In view of (4.5) we can now repeat the previous arguments to deduce that  $\nabla$  acts as  $\nabla \circ \Psi_\pm^* \subset \Psi_\pm^*$ , like the exterior derivative in Proposition 4.3. In particular the twisted Hodge Laplacian respects the even subcalculus as well. Note that unitarity of the representation  $\rho$  is irrelevant for (4.4) and (4.5); however, it is central for asserting that the twisted geometric Dirac operator are formally self-adjoint.

While the statements on evenness and oddness of the Hodge star operator  $*$  in [MAVE12] differ slightly from the discussion here, our arguments still imply that the Hodge Laplacian on an edge manifold with an admissible edge metric is even, as asserted in [MAVE12]. Evenness of the Hodge Laplacian as well as the fact that even kernels compose to even kernels again (and indeed form a subcalculus) is used in [MAVE12] to establish evenness of the heat kernel.

**Theorem 4.4** ([MAVE12]). *The heat kernel  $H_{\mathcal{F}}$  for the Friedrichs extension of the Hodge Laplacian on an incomplete edge space  $(M, g)$  with an admissible edge metric is an element of  $\Psi_+^{2,0,\mathcal{E}}$ , for some index set  $\mathcal{E}$  at rf and lf.*

We point out that evenness of  $H_{\mathcal{F}}$  was derived in [MAVE12] under the rescaling transformation  $\Phi$ . The precise relation between the asymptotics of Schwartz kernels with and without the rescaling has been worked out in [MAVE12, (3.13)] and the statement here follows by an easy counting of the leading orders in the front face asymptotics.

In view of the parity properties of the Gauss Bonnet and the odd signature operators, as established in Proposition 4.3, we introduce the notion of even and odd differential operators. Denote by  $\text{Diff}_e^q(M, {}^{ie}\Lambda^* T^* M \otimes E)$  the space of differential operators acting on sections of  ${}^{ie}\Lambda^* T^* M \otimes E$  by smooth linear combinations of  $\ell$ -th order compositions of edge vector fields  $\mathcal{V}_e$ .

**Definition 4.5.** An operator  $D \in x^{-q}\text{Diff}_e^q(M, {}^{ie}\Lambda^* T^* M \otimes E)$  is *even* if  $D \circ \Psi_\pm^*(M) \subset \Psi_\pm^*(M)$ . Similarly,  $D \in x^{-q}\text{Diff}_e^q(M, {}^{ie}\Lambda^* T^* M \otimes E)$  is *odd* if  $D \circ \Psi_\pm^*(M) \subset \Psi_\mp^*(M)$ .

Following the microlocal heat kernel construction in [MAVE12], we may similarly construct the heat kernel for the square of even or odd allowable Dirac operators, thereby extending the statement of Theorem 4.4

**Theorem 4.6.** *Let  $D$  be an allowable essentially self-adjoint Dirac operator of either even or odd parity on an incomplete edge space  $(M, g)$  with an admissible edge metric. Then the heat kernel of  $D^2$  is an element of  $\Psi_+^{2,0,\mathcal{E}}$ , for some index set  $\mathcal{E}$  at rf and lf.*

We point out that both the even and the odd operators are set to preserve the parity of coefficients in the asymptotics at the temporal diagonal td. This is in fact a general property of a Dirac type operator,  $D$ , well known in the classical setting of smooth manifolds. There, indeed, it is known, see for example [Gil84, Lemma 1.9.1], that in the asymptotic expansion of the pointwise trace

$$(4.6) \quad \text{tr} D e^{-tD^2}(p, p) \sim_{t \rightarrow 0+} \sum_{n=0}^{\infty} \kappa_n(p) t^{\frac{-m-1+n}{2}}$$

the even coefficients  $\kappa_{2k}$  vanish identically. In the singular setting this property of the expansion corresponds to the Schwartz kernel of  $D e^{-tD^2}$  having an even asymptotics at the temporal diagonal.

**4.2. The even and odd heat calculus for the spin Laplacian.** We assume that  $M$  is spin and consider the spinor bundle  $S$ . As before the heat calculus is defined as follows.

**Definition 4.7.** Let  $\mathcal{E} = (E_{\text{lf}}, E_{\text{rf}})$  be an index family for the two side faces of  $\mathcal{M}_h^2$ . We define  $\Psi_{\text{e-h}}^{\ell,p,\mathcal{E}}(M)$  to be the space of all operators  $P$  with Schwartz kernels  $K_P$  which are pushforwards from polyhomogeneous functions  $\beta^* K_P$  on  $\mathcal{M}_h^2$ , taking values in  $S \boxtimes S$  with index family  $(-m-2+\ell+\mathbb{N}_0, 0)$  at ff,  $(-m+p+\mathbb{N}_0, 0)$  at td,  $\emptyset$  at tf and  $\mathcal{E}$  for the two side faces of  $\mathcal{M}_h^2$ .

Following the even/odd classification introduced in the previous section, we now define the even/odd calculus by imposing conditions on the asymptotic expansions of elements in  $\Psi_{\text{e-h}}^*$  at the temporal diagonal td and the front face ff.

**Definition 4.8.** Consider  $P \in \Psi_{\text{e-h}}^*(M)$  with the Schwartz kernel  $K_P$ . Then  $P$  is said to be even if the following two conditions hold. At td we require in terms of the projective coordinates (3.4), valid in the interior of td

$$\beta^* K_P \sim \eta^{-m-2} \sum_{n=0}^{\infty} \kappa_n^{\text{td}}(S, U, Z, \tilde{x}, \tilde{y}, \tilde{z}) \eta^n,$$

$$\text{with } \kappa_n^{\text{td}}(-S, -U, -Z, \tilde{x}, \tilde{y}, \tilde{z}) = (-1)^n \kappa_n^{\text{td}}(S, U, Z, \tilde{x}, \tilde{y}, \tilde{z}) \text{ for all } n.$$

At ff we require in terms of the projective coordinates (3.2), valid in the interior of ff

$$\beta^* K_P \sim \rho^{-m-2} \sum_{n=0}^{\infty} \kappa_n^{\text{ff}}(\xi, \tilde{\xi}, u, z, \tilde{y}, \tilde{z}) \cdot (s_n, s'_n) \rho^n,$$

where the coefficients  $\kappa_n^{\text{ff}}$  are set to be scalar, and  $(s_n, s'_n)$  is a pair of spinors forming a section in  $S \boxtimes S$ . We denote the canonical inner product on fibres of  $S$  by  $(\cdot, \cdot)_S$  and

require the following evenness condition for the coefficients

$$\begin{aligned}\kappa_n^{\text{ff}}(\xi, \tilde{\xi}, -u, z, \tilde{y}, \tilde{z}) &= (-1)^n \kappa_n^{\text{ff}}(\xi, \tilde{\xi}, u, z, \tilde{y}, \tilde{z}), \text{ if } \text{tr}(s_n, s'_n) = (s_n, s'_n)_S \in C_+^\infty(M) \\ \kappa_n^{\text{ff}}(\xi, \tilde{\xi}, -u, z, \tilde{y}, \tilde{z}) &= (-1)^{n+1} \kappa_n^{\text{ff}}(\xi, \tilde{\xi}, u, z, \tilde{y}, \tilde{z}), \text{ if } \text{tr}(s_n, s'_n) = (s_n, s'_n)_S \in C_-^\infty(M)\end{aligned}$$

We denote the set of all even operators by  $\Psi_+^*(M)$ . We define the odd subcalculus  $\Psi_-^*(M)$  by replacing  $(-1)^n$  in the parity conditions at ff by  $(-1)^{n+1}$ . The parity at td remains unchanged. Note that the definition of  $\Psi_+^*(M)$  ensures the same asymptotics (4.3) as before in the setting of the Hodge Laplacian.

As in the previous subsection we classify even and odd edge differential operators.

**Definition 4.9.** An operator  $D \in x^{-q}\text{Diff}_e^q(M, S)$  is *even* if  $D \circ \Psi_\pm^*(M) \subset \Psi_\pm^*(M)$ . Similarly,  $D \in x^{-q}\text{Diff}_e^q(M, S)$  is *odd* if  $D \circ \Psi_\pm^*(M) \subset \Psi_\mp^*(M)$ .

We can now prove the following central theorem.

**Theorem 4.10.** *Assume that the spin Dirac operator  $D$  on an incomplete edge space  $(M, g)$  with an admissible edge metric satisfies the geometric Witt condition. Assume that for the dimension  $m$  of  $M$  and for the dimension  $b$  of any edge singularity, we always have  $(m - b)$  is even. Then  $D$  and  $D^2$  are both even operators and the heat kernel  $H$  for the spin Laplacian  $D^2$  is an element of  $\Psi_+^{2,0,\mathcal{E}}$  for some index set  $\mathcal{E}$  at rf and lf.*

*Proof.* Note that by the explicit formulae of [CHO85], the normal operator  $N(x^2 D^2)_{y_0}$  defines an initial heat kernel parametrix inside the even subcalculus  $\Psi_+^{2,0,\mathcal{E}}$ . If  $D$  maps the even subcalculus  $\Psi_+^*(M)$  to itself, then so does the spin Laplacian  $D^2$ . Following the argument outlined in [MAVE12], the exact heat kernel must be an element of the even subcalculus as well.

Hence it remains to study the action of  $D$  with respect to the even and odd subcalculi. Recall from above

$$(4.7) \quad D = c(\partial_x)\partial_x + \frac{f}{2x}c(\partial_x) + \frac{1}{x}\sum_{\alpha=1}^f c(x^{-1}V_\alpha)\nabla_{V_\alpha} + \sum_{\beta=1}^b c(U_\beta)\nabla_{U_\beta} + W,$$

where  $\{\partial_x, x^{-1}V_\alpha, U_\beta\}$  is an orthonormal frame of vector fields, which respects the splitting (3.5);  $\nabla$  the covariant derivative on the spin bundle, induced from the lift of the Levi-Civita connection to the spin principle bundle;  $c$  denotes the Clifford multiplication and  $W$  a higher order term, made explicit in the proof of [ALGE13, Lemma 1.1].

In order to clarify if  $D$  respects or interchanges the even and odd subcalculi, we need to make explicit how the pointwise trace on  $S \boxtimes S$  is induced by the admissible edge metric  $g$ . We also need to identify the Clifford action in explicit terms. Assume for the moment that  $m$  is even. Then [BGV04, Proposition 3.19] identifies the spinor bundle  $S$  locally as follows. Given an oriented local orthonormal basis<sup>9</sup>  $\{e_1, \dots, e_m\} := \{\partial_x, x^{-1}V_\alpha, U_\beta\}$  of  $TM$ , a polarization  $P$  of the complexified tangent bundle  $TM^\mathbb{C}$  is

<sup>9</sup>We assume for simplicity that the higher order term  $h$  in the Riemannian metric is zero.

defined in [BGV04, Definition 3.18] as the span of  $\{e_{2j-1} - ie_{2j} \mid j = 1, \dots, m/2\}$ . The spinor bundle  $S$  is then locally defined as the exterior algebra  $S = \Lambda^* P$ . The Riemannian metric  $g$ , extended to  $TM^{\mathbb{C}}$  by  $\mathbb{C}$ -linearity, now canonically defines an inner product on the fibres of  $S$ . The spinor bundle  $S$  decomposes into a direct sum of half spinor bundles  $S = S^+ \oplus S^-$ , where  $S^+ = \oplus_k \Lambda^{2k} P$  and  $S^- = \oplus_k \Lambda^{2k+1} P$ .

The Clifford multiplication is given under this identification as follows. The complexified tangent space  $TM^{\mathbb{C}}$  decomposes pointwise into a direct sum  $P \oplus \overline{P}$ . We may write any given tangent vector  $V$  as a direct sum of an element  $\pi(V)$  in  $P$  and an element  $\overline{\pi}(V)$  in the complex conjugate  $\overline{P} \cong P^*$ . Then for any  $s \in S$

$$(4.8) \quad \begin{aligned} c(\pi(V)) \cdot s &= \sqrt{2} \text{ext}(\pi(V))s, \\ c(\overline{\pi}(V)) \cdot s &= -\sqrt{2} \text{int}(\overline{\pi}(V))s. \end{aligned}$$

Now, if  $b$  is even, the fibre dimension  $f$  is odd and the polarization  $P$  is spanned by elements of the form  $\{\partial_x - ix^{-1}V_1, x^{-1}V_{2k} - x^{-1}V_{2k+1}, U_{2j-1} - iU_{2j} \mid k = 1, \dots, (f-1)/2, j = 1, \dots, b/2\}$ . As in the previous subsection for differential forms, we assign even parity to the vector fields  $U_\beta$ , and odd parity to vector fields  $\partial_x$  and  $x^{-1}V_\alpha$ . The crucial observation is that in the elements spanning  $P$  there are no pairing of even and odd vector fields, so that the spanning elements are either of even or of odd, but not of mixed parity. One checks explicitly that  $\pi(\partial_x) = (\partial_x - ix^{-1}V_1)/2$  and  $\overline{\pi}(\partial_x) = (\partial_x + ix^{-1}V_1)/2$  are both of odd parity. Similarly,  $\pi(V)$  and  $\overline{\pi}(V)$  are odd for  $V \in \{x^{-1}V_\alpha\}$ , and even for  $V \in \{U_\beta\}$ . Consequently, Clifford multiplication by odd vector fields is indeed odd, and Clifford multiplication by even vector fields is indeed even. Note that this line of argumentation breaks down completely if  $b$  is odd. In that case we cannot exclude elements of mixed parity from the set of generators of  $P$  and hence cannot classify the parity of the Clifford action any longer.

In view of the fact that in the spin Dirac operator action, the scalar action of each  $e_j$  is coupled with the Clifford action of  $c(e_j)$ , we conclude that leading order terms of the spin Dirac operator  $D$  are even if  $m$  is even and  $(m-b)$  is even. Evenness of the higher order terms  $W$  follows from their explicit structure, cf. the proof of [ALGE13, Lemma 1.1]. Now let us consider the case of  $m$  odd. In this case the spinor bundle arises locally as the half spinor bundle of  $M \times \mathbb{R}$ . The tangent vector coming from the additional  $\mathbb{R}$  component is assigned even parity. Repeating the arguments verbatim yields evenness of the spin Dirac operator  $D$  if  $(m-b)$  is even as well.  $\square$

We conclude the section with noting that the condition  $(m-b)$  to be even in the even odd classification of the signature and the spin Dirac operators, is naturally satisfied in the setting of algebraic varieties with edge singularities. This is of course due to the fact that all singularities must be of even real codimension,  $\mathbb{C}^n \cong \mathbb{R}^{2n}$ .

## 5. SHORT TIME ASYMPTOTIC EXPANSIONS OF THE HEAT TRACES

**5.1. Trace as an integral of the Schwartz kernel.** This subsection is concerned with establishing the trace class property for the operators  $e^{-tD^2}$  and  $De^{-tD^2}$  and deriving an integral formula for their traces, a version of Lidskii's theorem in this singular



setting. We present an argument that can then be adapted in the non-compact setting of Galois coverings below.

Let  $E$  be any Hermitian vector bundle and let, as usual,  $L^2(M, E)$  denote the space of square-integrable sections of  $E$  with an inner product defined by the Riemannian metric  $g$  on  $M$  and the Hermitian fibrewise metric  $h$  of  $E$ . We shall also consider the Hilbert space  $L^2(M \times M, E \boxtimes E^*)$ . We recall the following central definitions (cf. Shubin [Shu02, §2.20])

**Definition 5.1.** Let  $\{e_\alpha\}_{\alpha \in \mathcal{J}}$  denote some orthonormal basis of  $L^2(M, E)$  and write  $\|\cdot\|$  for the norm on  $L^2(M, E)$ . A bounded operator  $A$  on  $L^2(M, E)$  is said to be a Hilbert-Schmidt operator if

$$(5.1) \quad \sum_{\alpha \in \mathcal{J}} \|Ae_\alpha\|^2 < +\infty$$

We denote the set of Hilbert-Schmidt operators by  $\mathcal{C}_2(M, E)$ .

It is easy to see that the left hand side of (5.1) is independent of the choice of the orthonormal basis. The following are classic results:

- $\mathcal{C}_2(M, E)$  is an ideal;
- If  $K_A \in L^2(M \times M, E \boxtimes E^*)$  denotes the Schwartz kernel of a bounded operator  $A$  then  $A \in \mathcal{C}_2(M, E)$  if and only if  $K_A \in L^2(M \times M, E \boxtimes E^*)$ ;
- $\mathcal{C}_2(M, E)$  can be given the structure of a Hilbert space by defining the Hilbert-Schmidt inner product: if  $B, C \in \mathcal{C}_2(M, E)$  then

$$(5.2) \quad \langle B, C \rangle_2 := \iint_{M \times M} K_B(p, q) \overline{K_C(q, p)} \, d\text{vol}_g(q) d\text{vol}_g(p).$$

We denote the corresponding Hilbert Schmidt norm by  $\|A\|_2$

**Definition 5.2.** The ideal  $(\mathcal{C}_2(M, E))^2$  is, by definition, the ideal of trace class operators. Thus a bounded operator  $A$  is trace class if it is a finite linear combination of products of Hilbert-Schmidt operators. We denote the space of trace class operators by  $\mathcal{C}_1(M, E) \subset \mathcal{C}_2(M, E)$  and equip it with the trace norm  $\|A\|_1 := \text{Tr} |A|$ , where the trace functional  $\text{Tr}$  is defined on  $A = B \circ C$ ,  $B, C \in \mathcal{C}_2(M, E)$ , as

$$(5.3) \quad \begin{aligned} \text{Tr}(A) &:= \langle B, C^* \rangle_2 = \iint_{M \times M} K_B(p, q) \overline{K_{C^*}(q, p)} \, d\text{vol}_g(q) d\text{vol}_g(p) \\ &= \iint_{M \times M} K_B(p, q) K_C(q, p) \, d\text{vol}_g(q) d\text{vol}_g(p). \end{aligned}$$

**Remark 5.3.** The aim of this subsection is to establish an integral representation of the trace for the operators  $e^{-tD^2}$  and  $De^{-tD^2}$  in terms of their Schwartz kernels at the diagonal. While this might seem obvious by trying to replace the inner integral in (5.3) by  $K_A(p, p)$ , some care is necessary as explained e.g. in [Shu02, §2.21]. The important issue is that the Schwartz kernel of a trace class operator  $A = B \circ C$  with  $B, C \in \mathcal{C}_2(M, E)$  is given only almost everywhere by

$$(5.4) \quad K_A(p, q) = \int_M K_B(p, s) K_C(s, q) d\text{vol}_g(s).$$

In order to claim this identity at the set  $\{p = q\}$  of measure zero, one needs an additional assumption of continuity of  $K_A$  at the diagonal. Otherwise we cannot in general replace the inner integral in (5.3) by  $K_A(p, p)$  and present the trace of  $A$  solely by an integral of its Schwartz kernel over the diagonal.

We proceed by establishing the trace class property for  $e^{-tD^2}$  and  $De^{-tD^2}$ .

**Proposition 5.4.** *Let  $D$  be an allowable Dirac operator on an incomplete edge space  $(M, g)$ . Assume that  $D$  satisfies the geometric Witt condition. Let  $f \in \mathcal{S}(\mathbb{R})$  be a rapidly decreasing function and let  $f(D)$  the operator defined through functional calculus by the unique self-adjoint extension of  $D$ . Then  $f(D)$  and in particular  $e^{-tD^2}$  and  $De^{-tD^2}$  are trace class.*

*Proof.* We proceed by proving the statement first for a general  $f \in \mathcal{S}(\mathbb{R})$  and then provide an alternative argument for the particular examples  $e^{-tD^2}$  and  $De^{-tD^2}$ . Proceeding as in [ALMP13, Theorem 7.6] one proves that for any allowable Dirac operator satisfying the geometric Witt condition

$$(5.5) \quad (\text{Id} + D^2)^{-N} \text{ is trace class in } L^2$$

if  $N > \dim M/2$ . For these  $N$  we then have that  $f(D) = (1 + D^2)^{-N}((1 + D^2)^N f(D))$  is the composition of a bounded operator,  $(1 + D^2)^N f(D)$ , and of a trace class operator. Thus  $f(D)$  is trace class. For the particular examples  $e^{-tD^2}$  and  $De^{-tD^2}$  the trace class property can also be concluded by studying their Schwartz kernels as follows.

First of all let us consider the index set  $E = (E_{\text{rf}}, E_{\text{lf}})$  for the heat kernel of  $e^{-tD^2}$ , lifted to the heat space  $\mathcal{M}_h^2$ , at the right and left boundary faces. By symmetry of the heat kernel it suffices to discuss  $E_{\text{rf}}$  only. In view of the microlocal heat kernel construction as in [MAVE12], the index set can be written as  $E_{\text{rf}} = E_{\text{rf}}^0 + \mathbb{N}$ , where the integer numbers in  $\mathbb{N}$  are set to begin with 1, and  $E_{\text{rf}}^0 \geq 0$  is the index set of the initial parametrix defined by the heat kernel  $H_{\mathcal{C}(F)}$  of the model cone  $\mathcal{C}(F)$ . By construction, the asymptotic expansion of  $H_{\mathcal{C}(F)}$  at rf gets annihilated under the Laplacian  $\Delta_{\mathcal{C}(F)}$  and hence also under the Dirac operator  $N(D)$ , obtained by lifting  $D$  to  $\mathcal{M}_h^2$  and restricting its action to the front face ff. From here one easily concludes that the index set of  $De^{-tD^2}$  at rf is given by  $E'_{\text{rf}} = E_{\text{rf}}^0 + (\mathbb{N} - 1)$  and hence  $E'_{\text{rf}} = E_{\text{rf}} \geq 0$ .

From the microlocal description of the Schwartz kernel for  $e^{-tD^2}$  and  $De^{-tD^2}$ , lifted to the heat space  $\mathcal{M}_h^2$ , as well as from positivity of the index sets  $E = (E_{\text{rf}}, E_{\text{lf}})$  and  $E'_{\text{rf}}$ , we conclude that both Schwartz kernels are  $L^2$  integrable on  $M \times M$  for fixed times  $t > 0$ . We point out that for fixed  $t > 0$  we study the heat kernel asymptotics away from front face in  $\mathcal{M}_h^2$ . Consequently, the Schwartz kernels of  $e^{-tD^2}$  and  $De^{-tD^2}$  are  $L^2$  on  $M \times M$  and hence by Definition 5.1 the operators are Hilbert-Schmidt. Using this and the semigroup property we can write

$$(5.6) \quad \begin{aligned} De^{-tD^2} &= (De^{-\frac{t}{2}D^2})(e^{-\frac{t}{2}D^2}), \\ e^{-tD^2} &= (e^{-\frac{t}{2}D^2})(e^{-\frac{t}{2}D^2}). \end{aligned}$$

showing that the two operators are in fact trace class.  $\square$

Knowing that the operators  $e^{-tD^2}$  and  $De^{-tD^2}$  are trace class does not necessarily yield an integral formula for their traces in terms of their respective Schwartz kernels, as pointed out in Remark 5.3. However, by the microlocal construction of the heat kernel in §3 the Schwartz kernels of these operators are indeed smooth along the diagonal for each fixed  $t > 0$  so that we can conclude with the following central result.

**Proposition 5.5.** *Let  $D$  be an allowable Dirac operator on an incomplete edge space  $(M, g)$ . Assume that  $D$  satisfies the geometric Witt condition. Then the heat operator  $e^{-tD^2}$  for the unique self-adjoint extension of  $D^2$  at a fixed time  $t > 0$  is a trace class operator and for its trace the following formula holds:*

$$(5.7) \quad \text{Tr}(e^{-tD^2}) = \int_M \text{tr}_p \mathcal{H}(t)(p, p) d\text{vol}_g(p),$$

with  $\mathcal{H}(t)$  equal to the heat kernel. Similarly, at a fixed time  $t > 0$ , the operator  $De^{-tD^2}$  is trace class and its trace can be computed by the formula

$$(5.8) \quad \text{Tr}(De^{-tD^2}) = \int_M \text{tr}_p \mathcal{K}(t)(p, p) d\text{vol}_g(p),$$

where  $\mathcal{K}(t)$  equal to the Schwartz kernel of  $De^{-tD^2}$ .

*Proof.* By the microlocal description of the heat kernel in §3 the Schwartz kernels of  $e^{-tD^2}$  and  $De^{-tD^2}$  are smooth in  $M \times M$  and in particular they are continuous at the diagonal of  $M \times M$  and we can conclude<sup>10</sup> in view of (5.6) and (5.4) (we write  $dp = d\text{vol}_g(p)$  to simplify notation)

$$\begin{aligned} \text{Tr } De^{-tD^2} &= \int_M \int_M \left( De^{-\frac{t}{2}D^2} \right)(p, q) \left( e^{-\frac{t}{2}D^2} \right)(q, p) dq dp = \int_M \left( De^{-tD^2} \right)(p, p) dp \\ \text{Tr } e^{-tD^2} &= \int_M \int_M \left( e^{-\frac{t}{2}D^2} \right)(p, q) \left( e^{-\frac{t}{2}D^2} \right)(q, p) dq dp = \int_M \left( e^{-tD^2} \right)(p, p) dp \end{aligned}$$

In both identities the first equality follows from the semigroup property in (5.6) and Definition 5.1 of the trace, and the second identity follows from continuity of the Schwartz kernels at the diagonal, which yields an identity of the form (5.4) at  $\{p = q\}$ .  $\square$

We will now proceed with a derivation of an asymptotic expansion for the traces  $\text{Tr}(e^{-tD^2})$  and  $\text{Tr}(De^{-tD^2})$  for allowable even or odd Dirac operators  $D$  under the geometric Witt assumption.

**5.2. The push-forward theorem of Melrose and the asymptotic expansion of the heat traces.** Consider the diagonal  $D_0 \subset \mathbb{R}^+ \times M_c^2$  and its lift  $D_h$  to  $\mathcal{M}_h^2$ , as pictured in Fig. 2. Denote by  $D_{\text{td}} = D_h \cap \text{td}$ ,  $D_{\text{ff}} = D_h \cap \text{ff}$  and  $D_{\text{cf}} = D_h \cap \text{rf} \cap \text{lf}$  the boundary faces of the lifted diagonal  $D_h$ . Let  $\iota : D_h \hookrightarrow \mathcal{M}_h^2$  denote the natural embedding and  $\iota^*H$  the restriction of  $H \equiv e^{-tD^2}$  to the diagonal. Since  $H \in \Psi_+^{2,0,\mathcal{E}}(M)$ , the restriction  $\iota^*H$  and its pointwise trace  $\text{tr } H$  are polyhomogeneous on  $D_h$ . Polyhomogeneity is straightforward and can be checked in local coordinates, but is also

<sup>10</sup>The trace integral formula for  $e^{-tD^2}$  can be obtained differently from the trace-class property and positivity of the operator using Mercer theorem.

a special case of the ‘pullback theorem’ in [MEL92]. The index family of  $\text{tr } \iota^* H$  is given by

$$(5.9) \quad G_{\text{td}} = -m + 2\mathbb{N}_0, \quad G_{\text{ff}} = -m + 2\mathbb{N}_0, \quad G_{\text{cf}} = E_{\text{lf}} + E_{\text{rf}},$$

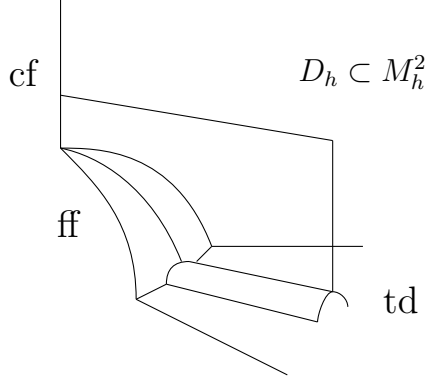


FIGURE 2. The diagonal hypersurface  $D_h$  in  $\mathcal{M}_h^2$ .

where the odd terms in  $G_{\text{td}}$  and  $G_{\text{ff}}$  vanish due to evenness conditions in Definitions 4.2 and 4.8. Note that functions on  $\text{td}$  which are odd with respect to the reflection  $(S, U, Z) \mapsto -(S, U, Z)$  in coordinates (3.4), vanish on  $D_h$  and do not contribute to the pointwise trace asymptotics. Similarly, functions on  $\text{ff}$  which are odd with respect to  $u \mapsto -u$  in coordinates (3.2), vanish on  $D_h$  as well and thus also have vanishing trace.

We now want to express the actual trace  $\text{Tr } H$  as a pushforward of some polyhomogeneous density. Recall here that by Proposition 5.5 the heat operator is trace class and that the trace  $\text{Tr } H$  is given by the integral of the pointwise trace of the heat kernel along the diagonal.

We employ a formalism for understanding pushforwards of polyhomogeneous distributions, known as Melrose’s Pushforward Theorem [MEL92]. Let the map  $\pi_c : D_h \rightarrow \mathbb{R}^+$  be defined as the composition of the blowdown map  $D_h \rightarrow \mathbb{R}^+ \times M_c$  and the projection  $\mathbb{R}^+ \times M_c \rightarrow \mathbb{R}^+$ . Then, following straightforward computations outlined in [MAVE12, §4],  $\text{Tr } H$  is the pushforward of  $\text{tr } H$  by  $\pi_c$ :

$$(5.10) \quad \text{Tr } H(t) \frac{dt}{t} = (\pi_c)_* (2\rho_{\text{ff}}\rho_{\text{cf}} \text{tr } H \beta^*(x^f) \mu_b),$$

where  $\mu_b$  is a  $b$ -density on  $D_h$  of the form  $(\rho_{\text{td}}\rho_{\text{ff}}\rho_{\text{cf}})^{-1}\mu_0$  for some density  $\mu_0$  which is smooth up to all boundary faces and nowhere vanishing. The factor  $\beta^*(x^f)$  comes from the volume form  $\text{dvol}(g)$  with equals  $x^f dx dy dz$  up to some bounded function on  $M$ . Most importantly, Melrose’s pushforward theorem asserts (cf. [MAVE12, (4.3)]) that for the index family  $\mathcal{G} = (G_{\text{td}}, G_{\text{ff}}, G_{\text{cf}})$  of  $\rho_{\text{ff}}\rho_{\text{cf}} \text{tr } H_{\mathcal{F}}$ , the index family of the pushforward  $\text{Tr } H(t)$  is given by  $F = \pi_c^b(\mathcal{G})$  which is defined as follows. If  $G_{\text{td}} = \{(z_j, p_j)\}$  and  $G_{\text{ff}} = \{(w_\ell, q_\ell)\}$ , then  $G'_{\text{td}} = \{(z_j/2, p_j)\}$ ,  $G'_{\text{ff}} = \{(w_\ell/2, q_\ell)\}$  and  $F = G'_{\text{td}} \cup G'_{\text{ff}}$  is the ‘extended union’ of  $G'_{\text{td}}$  and  $G'_{\text{ff}}$

$$(5.11) \quad G'_{\text{td}} \cup G'_{\text{ff}} = G'_{\text{td}} \cup G'_{\text{ff}} \cup \{(z, p+q+1) : \exists (z, p) \in G'_{\text{td}}, \text{ and } (z, q) \in G'_{\text{ff}}\}.$$

We refer to [MEL92, (42)] for a proof. Evaluating the index sets explicitly from (5.9), we obtain in view of (5.11)

$$(5.12) \quad \text{Tr } H(t) \sim \sum_{\ell=0}^{\infty} U_{\ell} t^{\ell-\frac{m}{2}} + \sum_{\ell=0}^{\infty} V_{\ell} t^{\ell-\frac{b}{2}} + \sum_{\ell \in \mathfrak{J}} W_{\ell} t^{\ell-\frac{b}{2}} \log t,$$

where  $\mathfrak{J} = \emptyset$  if  $(m-b)$  is odd and  $\mathfrak{J} = \mathbb{N}_0$  if  $(m-b)$  is even. Here, the coefficients  $U_{\ell}$  are defined as integrals over  $M$  of pointwise traces of coefficients in the asymptotic expansion of  $\beta^* H$  at  $\text{td}$ . These coefficients are functions on  $M \times M$  and their pointwise traces are functions over  $M$ , depending pointwise on the jets of the Riemannian metric and only on the rank of the twisting vector bundle. The coefficients  $V_{\ell}$  and  $W_{\ell}$  are defined as integrals over  $\partial M$  of pointwise traces of coefficients in the product type asymptotic expansion at the front face intersecting  $\text{td}$ . These coefficients are functions over  $\partial M \times \partial M$  and their pointwise traces are functions over  $\partial M$ , depending pointwise on the jets of the Riemannian metric and only on the rank of the twisting vector bundle. All of this is explained in detail in §5.3.

A similar analysis may be performed for the trace class operator  $\text{Tr}(DH)$ .

**Theorem 5.6.** *Let  $D$  be an even or odd essentially self-adjoint allowable Dirac operator on an admissible edge manifold  $(M, g)$  with  $\dim M = m$  and edge  $B$  of dimension  $b$ . Then*

$$(5.13) \quad \begin{aligned} \text{Tr } DH &\sim_{t \rightarrow 0} \sum_{\ell=0}^{\infty} A_{\ell} t^{\ell-\frac{m}{2}} + \sum_{\ell=0}^{\infty} B_{\ell} t^{\ell-\frac{b}{2}} + \sum_{\ell \in \mathfrak{J}} C_{\ell} t^{\ell-\frac{b}{2}} \log t, \text{ if } D \text{ is even,} \\ \text{Tr } DH &\sim_{t \rightarrow 0} \sum_{\ell=0}^{\infty} A_{\ell} t^{\ell-\frac{m}{2}} + \sum_{\ell=0}^{\infty} B_{\ell} t^{\ell-\frac{b+1}{2}} + \sum_{\ell \in \mathbb{N}_0 \setminus \mathfrak{J}} C_{\ell} t^{\ell-\frac{b+1}{2}} \log t, \text{ if } D \text{ is odd,} \end{aligned}$$

where in both cases  $\mathfrak{J} = \emptyset$  if  $(m-b)$  is odd and  $\mathfrak{J} = \mathbb{N}_0$  if  $(m-b)$  is even.

*Proof.* Let  $D$  be even. Evenness of  $D$  implies by definition  $DH \in \Psi_+^{1,-1,\mathcal{E}-1}(M)$ . Consequently, the pointwise trace  $\text{tr } DH$  lifts to a polyhomogeneous function on the diagonal  $D_h$  with the index sets

$$(5.14) \quad \begin{aligned} G_{\text{td}} &= (-1 - m + \mathbb{N}_0) \cap (-2 - m + 2\mathbb{N}_0) = -m + 2\mathbb{N}_0, \\ G_{\text{ff}} &= (-1 - m + \mathbb{N}_0) \cap (-2 - m + 2\mathbb{N}_0) = -m + 2\mathbb{N}_0, \\ G_{\text{cf}} &= E_{\text{lf}} + E_{\text{rf}} - 1 > -1, \end{aligned}$$

Note that  $G_{\text{td}}$  corresponds to the interior expansion in (4.6). Let now  $D$  be odd, so that by definition  $DH \in \Psi_-^{1,-1,\mathcal{E}-1}(M)$ . Consequently, the pointwise trace  $\text{tr } DH$  lifts to a polyhomogeneous function on the diagonal  $D_h$  with the index sets

$$(5.15) \quad \begin{aligned} G_{\text{td}} &= (-1 - m + \mathbb{N}_0) \cap (-2 - m + 2\mathbb{N}_0) = -m + 2\mathbb{N}_0, \\ G_{\text{ff}} &= (-1 - m + \mathbb{N}_0) \cap (-1 - m + 2\mathbb{N}_0) = -1 - m + 2\mathbb{N}_0, \\ G_{\text{cf}} &= E_{\text{lf}} + E_{\text{rf}} - 1 > -1, \end{aligned}$$

The statement now follows by making the index set in (5.11) explicit.  $\square$

In the special case of  $D$  being the Gauss-Bonnet, the signature or the spin Dirac operator, we can obtain a much stronger statement.

**Theorem 5.7.** *Let  $D$  be an essentially self-adjoint geometric Dirac operator, i.e. either the Gauss-Bonnet, the signature or the spin Dirac operator on an admissible edge manifold  $(M, g)$  with  $\dim M = m$  and edge  $B$  of dimension  $b$ . Then  $\text{Tr } DH \equiv 0$  if  $m$  is even, and the following short time asymptotic expansion holds if  $m$  is odd*

$$\text{Tr } DH \sim_{t \rightarrow 0} \sum_{\ell=\frac{(m+1)}{2}}^{\infty} A_{\ell} t^{\ell-\frac{m}{2}} + \sum_{\ell=0}^{\infty} B_{\ell} t^{\ell-\frac{b}{2}} + \sum_{\ell \in \mathfrak{I}} C_{\ell} t^{\ell-\frac{b}{2}} \log t, \text{ if } D \text{ is even.}$$

Note that the class of even geometric Dirac operators  $D$  on odd-dimensional manifolds includes the Gauss-Bonnet operator for any  $b$ , and the odd signature as well as the spin Dirac operator for  $(m-b)$  even.

$$\text{Tr } DH \sim_{t \rightarrow 0} \sum_{\ell=\frac{(m+1)}{2}}^{\infty} A_{\ell} t^{\ell-\frac{m}{2}} + \sum_{\ell=0}^{\infty} B_{\ell} t^{\ell-\frac{b+1}{2}} + \sum_{\ell \in \mathbb{N}_0 \setminus \mathfrak{I}} C_{\ell} t^{\ell-\frac{b+1}{2}} \log t, \text{ if } D \text{ is odd.}$$

The class of odd geometric Dirac operators on odd dimensional manifolds includes the odd signature operator for  $(m-b)$  odd. In both cases  $\mathfrak{I} = \emptyset$  if  $(m-b)$  is odd and  $\mathfrak{I} = \{\ell \in \mathbb{N}_0 \mid \ell \geq \frac{(m+1)}{2}\}$  if  $(m-b)$  is even.

*Proof.* Let  $E$  denote  $S \otimes F$  in case  $D$  refers to the spin Dirac operator, and  $\Lambda^* T^* M \otimes F$  in case  $D$  refers to the Gauss-Bonnet or the signature operator. Here  $F$  is any flat Hermitian vector bundle induced by a unitary representation, and  $S$  the spinor bundle. Assume first that  $m$  is even. Then, exactly as in the smooth compact setting, there exists a unitary  $\phi : L^2(M, E) \rightarrow L^2(M, E)$ , respecting the self-adjoint domain of  $D$  such that  $\phi D + D\phi = 0$ . Consequently, the spectrum of  $D$  is symmetric around zero and we conclude  $\text{Tr } DH = 0$ .

Assume now that  $m$  is odd. Then the spectrum of  $D$  need not be symmetric any longer and the trace asymptotics is non-trivial. Note that each coefficient  $A_{\ell}$  comes from the heat kernel expansion at  $\text{td}$ , and hence is local in the sense that  $A_{\ell} = \int_M a_{\ell}$ , where at any given point  $p \in M$  each  $a_{\ell}(p)$  depends only on the jets of the Riemannian metric tensor  $g$  at  $p$ . Using Getzler rescaling and local index theorem techniques, Bismut and Freed [BiFr86] have established vanishing of the local coefficients  $a_{\ell}$  for  $(2\ell - m) < 1$ , compare also Melrose [MEL93]. Consequently,  $A_{\ell}$  vanish for  $(2\ell - m) < 1$ .

Now, by the definition (5.11) of extended unions, coefficients  $C_{\ell}$  arise if and only if the asymptotic expansion of  $\text{Tr } DH$  admits terms of the form  $A_{\ell+\frac{f}{2}}$  and  $B_{\ell}$  are both non-zero. In particular,  $C_{\ell}$  vanish for  $(2\ell - m) < 1$  as well. This proves the statement.  $\square$

We point out that without the assumption of  $D$  being even and odd, all statements in Theorems 5.6 and 5.7 continue to hold with the index  $\ell$  being fractional,  $\ell \in \mathbb{N}_0/2$ .

**5.3. On locality of coefficients in the asymptotic expansions.** In this subsection we clarify to what extent the coefficients  $A_{\ell}, B_{\ell}$  and  $C_{\ell}$  for  $\ell \in \mathbb{N}_0$  in the various asymptotic expansions of Theorems 5.6 and 5.7 depend on the choice of a twisting flat



vector bundle  $E$ . Consider  $\iota^*(DH)$ , its asymptotics at the temporal diagonal and its product type asymptotic expansion near the lower corner of the front face in  $D_h$

$$(5.16) \quad \begin{aligned} \iota^*(DH) &\sim \sum a_\ell \rho_{\text{td}}^{-m-1+\ell}, \quad \rho_{\text{td}} \rightarrow 0, \\ \iota^*(DH) &\sim \sum b_{kj} (\rho_{\text{td}} \rho_{\text{ff}})^{-m-1} \rho_{\text{td}}^k \rho_{\text{ff}}^j, \quad \rho_{\text{ff}}, \rho_{\text{td}} \rightarrow 0, \end{aligned}$$

where the coefficients  $a_\ell$  are smooth sections of  $({}^{ie}\Lambda^* T^* M \otimes E) \boxtimes ({}^{ie}\Lambda^* T^* M \otimes E)$  in the Hodge de Rham setting, and sections of  $(S \otimes E) \boxtimes (S \otimes E)$  in the spin setting. The coefficients  $b_{kj}$  are smooth sections of the same vector bundles restricted to  $\partial M$ . Taking pointwise traces, the coefficients  $\text{tr } a_\ell$  are smooth scalar functions on  $M$ , while the coefficients  $\text{tr } b_{jk}$  are smooth scalar functions on  $\partial M$ .

Short time asymptotics of  $\text{Tr } DH$  follows from either an application of the Pushforward theorem by Melrose or in this specific situation by direct computations from the pointwise asymptotic expansions in (5.16). In either approach it is clear that the coefficients  $A_\ell$  are given as integrals over  $M$  of  $\text{tr } a_\ell$ , whereas the coefficients  $B_\ell$  and  $C_\ell$  are induced by the terms  $\text{tr } b_{jk}$ , which are functions over  $\partial M$  and are consequently given as integrals over  $\partial M$ . Thus, the heat kernel expansion at  $\text{td}$  contributes coefficients to the short time asymptotics of  $\text{Tr } DH$  that are defined as integrals over  $M$ . At the same time, the heat kernel expansion at  $\text{ff}$  contributes coefficients that are defined as integrals over the measure zero subset  $\partial M$ .

In view of Theorem 3.4, the pointwise traces  $\text{tr } a_\ell$  and  $\text{tr } b_{jk}$  do not depend on the choice of  $E$ , but only on its rank. This proves the following theorem

**Theorem 5.8.** *Consider the short time asymptotic expansions in Theorems 5.6 and 5.7. Their coefficients are given by integrals of local quantities in the sense that  $A_\ell = \int_M a_\ell$ ,  $B_\ell = \int_{\partial M} b_\ell$  and  $C_\ell = \int_{\partial M} c_\ell$ , where at any given point  $p$  the values  $a_\ell(p)$ ,  $b_\ell(p)$  and  $c_\ell(p)$  depend only on the jets of the Riemannian metric tensor  $g$  at  $p$  and are in fact independent of the choice of  $E$  up to its rank.*

**5.4. Heat kernel on iterated cone-edge spaces.** We remark that the microlocal description of the heat kernel carries over to the case of  $D$  being a geometric Dirac operator on the higher depth iterated cone-edge spaces singled out in §2.2. More precisely, we assume that the bottom statum, the most singular, is a point  $P$  and the Riemannian metric in a neighbourhood of  $P$  takes the form  $g \upharpoonright \mathcal{U} = dx^2 + x^2 \kappa$  where the link  $(F, \kappa)$  is a stratified space with an iterated cone-edge metric  $\kappa$  with the property that  $D^2$  is the Hodge de Rham or the spin Laplacian and the tangential operator  $D_F$  admits discrete spectrum (the Gauss Bonnet or the signature operator associated to a Witt space  $F$  of arbitrary depth with a rigid iterated edge metric would be an example).

In this case, we may employ the discrete spectral decomposition of  $D_F$  on the cross section  $F$  to write the heat kernel of  $D^2$  explicitly in terms of Bessel functions. The corresponding formula in the Hodge de Rham setting is provided in [MAVE12, (3.9)]. The corresponding formula in the spin setting is provided above in (3.8). Since the Riemannian metric is assumed to be exact near the cone tip  $P$ , one does not require any microlocal calculus techniques to construct an exact heat kernel solution and may infer the heat kernel asymptotics straight from the explicit formulae. In particular,

(5.12) continues to hold in that specific iterated cone-edge setting for  $D^2$  being the Hodge de Rham or the spin Laplace operator. Note that due to the explicit formulae, this cone edge singularity, the point  $P$ , contributes only the  $B_0$  and does not contribute any  $C_*$  coefficients.

However an extension of an even and odd classification of geometric operators and definition of an even and odd subcalculus as in §4, to higher depth iterated cone-edge space is not obvious. While we may still try to mimic the even and odd subcalculus conditions at the front face  $\text{ff}$ , an extension of Theorems 5.6 and 5.7 depends on the explicit action of  $D$  on the eigenspaces of the tangential operator  $D_F$ , which is now complicated due to  $F$  being singular.

## 6. EXISTENCE OF ETA INVARIANTS ON SINGULAR EDGE SPACES

We are now in the position to discuss existence of eta invariants in various configurations involving admissible edge singularities. We prove existence of eta invariants for allowable essentially self-adjoint even or odd Dirac operators  $D$  in the incomplete admissible edge setting. Essential self-adjointness is guaranteed by the geometric Witt condition we have imposed throughout the paper.

In view of the asymptotic expansions obtained in the previous section, as well as exponential decay of the heat kernel for large times, the eta-function

$$\eta(D, s) = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2} \text{Tr}(De^{-tD^2}) dt$$

extends to a meromorphic function on  $\mathbb{C}$  with the residue at  $s = 0$  given by the  $t^{-\frac{1}{2}}$  coefficient in the short time asymptotics of  $\text{Tr}(De^{-tD^2})$ , up to some universal proportionality factor. In particular, if the asymptotic expansion does not admit  $t^{-\frac{1}{2}}$  terms, the eta invariant  $\eta(D) := \eta(D, s = 0)$  is well-defined. Hence just by looking at the asymptotics in Theorem 5.6 and studying contributions to the  $t^{-\frac{1}{2}}$  and the  $t^{-\frac{1}{2}} \log t$  coefficients we obtain the following a priori statement.

**Proposition 6.1.** *Let  $(M, g)$  be an incomplete edge space with an admissible edge metric. Let  $D$  be an allowable even or odd essentially self-adjoint Dirac operator. Then*

- (i) *the eta invariant  $\eta(D)$  is well-defined<sup>11</sup> if  $m$  is even, and  $(D, b)$  are either both of even or both of odd parity.*
- (ii) *the residue of  $\eta(D, s)$  at  $s = 0$  comes from the interior only<sup>12</sup>, i.e. is a local contribution, if  $m$  is odd, and  $(D, b)$  are either both of even or both of odd parity.*
- (iii) *the residue of  $\eta(D, s)$  at  $s = 0$  comes from the interior and from the edge fibration<sup>13</sup>, if  $D$  is even and either  $(m - b)$  is odd or  $b$  is even; or if  $D$  is odd and either  $(m - b)$  is even or  $b$  is odd.*

<sup>11</sup>in these cases there are no  $t^{-1/2}$  coefficients in the asymptotics of  $\text{Tr}(DH)$ .

<sup>12</sup>in these cases there are no  $B_* t^{-1/2}$  coefficients in the asymptotics of  $\text{Tr}(DH)$ .

<sup>13</sup>in these cases there are no  $C_* t^{-1/2}$  coefficients in the asymptotics of  $\text{Tr}(DH)$ .

- (iv) the eta function  $\eta(D, s)$  may admit a second order pole singularity<sup>14</sup> at  $s = 0$  if  $D$  is even,  $(m - b)$  is even and  $b$  is odd; or if  $D$  is odd,  $(m - b)$  is odd and  $b$  is even.

In the special case of  $D$  being the Gauss-Bonnet, the signature or the spin Dirac operator, we employ Theorem 5.7 to obtain a much stronger statement.

**Proposition 6.2.** *Let  $(M, g)$  be an incomplete edge space with an admissible edge metric. Let  $D$  be an essentially self-adjoint geometric Dirac operator, i.e. either the Gauss-Bonnet, the signature or the spin Dirac operator. Then*

- (i) *the eta invariant  $\eta(D)$  is identically zero if  $m$  is even.*
- (ii) *the eta invariant  $\eta(D, s)$  has at most a first order pole singularity at  $s = 0$ , if  $m$  is odd. Then the residue at  $s = 0$  always comes from the edge fibration only, i.e. is given by an integral over  $\partial M$  of a quantity that depends locally on the fibration  $\phi : \partial M \rightarrow B$  and the asymptotics of the Riemannian metric  $g$  at the edge.*

*Proof.* The statement follows directly from Theorem 5.7. In case  $m$  is even, triviality of the eta function is a consequence of the spectrum being symmetric around zero. In case  $m$  is odd, we just observe that the non-trivial contribution to the residue comes only from  $B_\ell$  coefficients that are integrals over  $\partial M$  of quantities that depend locally on the fibration  $\phi : \partial M \rightarrow B$  and the asymptotics of the Riemannian metric  $g$  at the edge. We may try to exclude the  $B_* t^{-1/2}$  coefficient from the asymptotics of  $\text{Tr } DH$  by asking  $(D, b)$  to be of the same parity. However, on an odd-dimensional manifold, the geometric Dirac operators are odd if  $b$  is even and vice versa. Hence the residue is a priori non-zero.  $\square$

## 7. ETA INVARIANTS ON GALOIS COVERINGS OF ADMISSIBLE EDGE SPACES

**7.1. Geometric preliminaries.** Let  $\overline{M}$  be a smoothly stratified pseudomanifold of arbitrary depth. Consider a Galois covering  $\pi : \overline{M}_\Gamma \rightarrow \overline{M}$  with Galois group  $\Gamma$  and fundamental domain  $\mathcal{F}_\Gamma$ . There is a natural way to define a topological stratification on  $\overline{M}_\Gamma$ . Decompose  $\overline{M}_\Gamma$  into the preimages under  $\pi$  of the strata in  $\overline{M}$ . Surjectivity of  $\pi$  ensures that each stratum in the covering is mapped surjectively onto the corresponding stratum in  $\overline{M}$ . Since  $\pi$  is a local homeomorphism, it is straightforward to check that  $\overline{M}_\Gamma$  and its fundamental domain are again topological stratified spaces.

In fact, more is true: by using these local homeomorphisms we can induce a *smooth* stratification on  $\overline{M}_\Gamma$  by simply pulling it up from the base and this argument applies in the Whitney as well as the Thom-Mather cases. It is important to point out that, by definition, the link of a point  $\tilde{p} \in \overline{M}_\Gamma$  is equal to the link of its image in the base. Needless to say, if  $\overline{M}_\Gamma$  is the universal covering space of  $\overline{M}$ , the individual strata in  $\overline{M}_\Gamma$  need not be the universal covering of the corresponding strata in the base.

We shall apply these remarks to a depth-one smoothly stratified space. We denote by  $\widetilde{M}$  the regular stratum of  $\overline{M}_\Gamma$  and observe that it is a Galois covering of the regular stratum  $M$  of  $\overline{M}$  with fundamental domain  $\mathcal{F}$  equal to the regular part of  $\mathcal{F}_\Gamma$ . Let  $g$

<sup>14</sup>in these cases we cannot exclude  $C_* t^{-1/2}$  coefficients in the asymptotics of  $\text{Tr}(DH)$ .

be an admissible incomplete edge metric on  $M$ . We can lift  $g$  to the Galois covering  $\widetilde{M}$  where it becomes a  $\Gamma$ -invariant admissible incomplete edge metric  $\widetilde{g}$ . Moreover, there is an isometric embedding of  $\mathcal{F}$  into  $M$  with complement of measure zero.

**7.2.  $\Gamma$ -Hilbert Schmidt and  $\Gamma$ -trace class operators.** In this section we introduce the notion of  $\Gamma$ -Hilbert Schmidt and  $\Gamma$ -trace class operators in full analogy to Definition 5.1, carried over to the present non-compact setting of Galois coverings. Our primary sources here are Shubin [SHU02] and Atiyah [ATI76].

We first recall the notion of bounded equivariant linear operators on  $L^2(\widetilde{M})$ . Let  $\mathcal{N}_\Gamma$  be the Von Neumann algebra of the discrete group  $\Gamma$ ; thus, by definition,  $\mathcal{N}_\Gamma$  is the weak closure of the algebra  $\mathbb{C}\Gamma$  viewed as an algebra of bounded operators on  $\ell^2(\Gamma)$  by convolution. There is a finite normal trace on  $\mathcal{N}_\Gamma$  defined as  $\tau_\Gamma(A) = (A\delta_e, \delta_e)_{\ell^2}$ .

The von Neumann algebra  $\mathcal{A}_\Gamma(\widetilde{M})$  associated to the  $\Gamma$ -Galois covering  $\widetilde{M}$  is defined, as usual, as the algebra of bounded equivariant linear operators on  $L^2(\widetilde{M})$ . As in the closed case there is an isomorphism  $L^2(\widetilde{M}) \simeq L^2(F) \otimes \ell^2(\Gamma) \equiv L^2(M) \otimes \ell^2(\Gamma)$ . Through this isomorphism the Von Neumann algebra  $\mathcal{A}_\Gamma(\widetilde{M})$  can be shown to be naturally isomorphic to  $\mathcal{B}(L^2(M)) \otimes \mathcal{N}_\Gamma$  and we can define through this isomorphism a semifinite trace, denoted  $\text{Tr}_\Gamma$ , by considering the tensor product  $\text{Tr} \otimes \tau_\Gamma$ . Here,  $\mathcal{B}(L^2(M))$  denotes the space of bounded operators. Similar definitions can be given when there is a vector bundle  $Q$  on  $M$  or a  $\Gamma$ -equivariant vector bundle  $\widetilde{Q}$  on  $\widetilde{M}$ . We could now go on and give, abstractly, definitions for the ideal of  $\Gamma$ -Hilbert-Schmidt and  $\Gamma$ -trace class operators.

However, in order to keep the theory of Von Neumann algebras to a minimum, we prefer to use as definitions what are in fact characterizations of  $\Gamma$ -Hilbert Schmidt and  $\Gamma$ -trace class operators, using Shubin [SHU02, §2.23 Theorems 1. and 3.] and Atiyah [ATI76, §4].

**Definition 7.1.** Consider a  $\Gamma$ -equivariant Hermitian vector bundle  $\widetilde{Q}$  on  $\widetilde{M}$ . Denote the space of square integrable sections of  $\widetilde{Q}$  by  $L^2(\widetilde{M}, \widetilde{Q})$ . Consider the von Neumann algebra  $\mathcal{A}_\Gamma(\widetilde{M}, \widetilde{Q})$  of bounded equivariant linear operators on  $L^2(\widetilde{M}, \widetilde{Q})$ . Then an operator  $A \in \mathcal{A}_\Gamma(\widetilde{M}, \widetilde{Q})$  with Schwartz kernel  $K_A$  is called

- (i) a  $\Gamma$ -Hilbert-Schmidt operator if  $K_A \in L^2(\widetilde{M} \times \mathcal{F}, \widetilde{Q} \boxtimes \widetilde{Q}^*)$ , or equivalently if  $\phi A$  and  $A\phi$  are Hilbert Schmidt in the sense of Definition 5.1 for any bounded measurable function  $\phi$  with compact support in the closure of  $\widetilde{M}$ . We denote the space of Hilbert-Schmidt operators by  $\mathcal{C}_2^\Gamma(\widetilde{M}, \widetilde{Q})$ . One can show that  $\mathcal{C}_2^\Gamma(\widetilde{M}, \widetilde{Q})$  is an ideal in  $\mathcal{A}_\Gamma(\widetilde{M}, \widetilde{Q})$ .

- (ii) a  $\Gamma$ -trace class operator if  $A = \sum B_j \circ C_j$  (finite sum) with  $B_j, C_j \in \mathcal{C}_2^\Gamma(\widetilde{M}, \widetilde{Q})$ . We denote the space of  $\Gamma$ -trace class operators by  $\mathcal{C}_1^\Gamma(\widetilde{M}, \widetilde{Q}) \subset \mathcal{C}_2^\Gamma(\widetilde{M}, \widetilde{Q})$ . Thus

$$\mathcal{C}_1^\Gamma(\widetilde{M}, \widetilde{Q}) := (\mathcal{C}_2^\Gamma(\widetilde{M}, \widetilde{Q}))^2 \subset \mathcal{C}_2^\Gamma(\widetilde{M}, \widetilde{Q})$$

We define for any  $A \in \mathcal{C}_1^\Gamma(\widetilde{M}, \widetilde{Q})$  the  $\Gamma$ -trace in terms of the characteristic function  $\phi$  of the fundamental domain  $\mathcal{F}$  by

$$(7.1) \quad \mathrm{Tr}_\Gamma(A) := \mathrm{Tr}(\phi A \phi) = \iint_{\mathcal{F} \times \widetilde{M}} K_B(p, q) K_C(q, p) \, \mathrm{dvol}_{\widetilde{g}}(q) \, \mathrm{dvol}_{\widetilde{g}}(p).$$

These definitions are independent of the choice of the fundamental domain  $\mathcal{F}$ .

**Remark 7.2.** As previously, one might want to replace the inner integral in (7.1) by  $K_A(p, p)$  thus obtaining the  $\Gamma$ -trace of  $A$  by integrating its Schwartz kernel at the diagonal over the fundamental domain  $\mathcal{F}$ :

$$(7.2) \quad \mathrm{Tr}_\Gamma(A) := \mathrm{Tr}(\phi A \phi) = \int_{\mathcal{F}} K_A(p, p) \, \mathrm{dvol}_{\widetilde{g}}(p)$$

However, the Schwartz kernel of  $A = B \circ C$  with  $B, C \in \mathcal{C}_2(M, E)$  is given only almost everywhere by

$$(7.3) \quad K_A(p, q) = \int_{\widetilde{M}} K_B(p, s) K_C(s, q) \, \mathrm{dvol}_{\widetilde{g}}(s).$$

In order to claim this identity at the set  $\{p = q\}$  of measure zero, one needs an additional assumption of continuity of  $K_A$  at the diagonal. Otherwise we cannot generally replace the inner integral in (7.1) by  $K_A(p, p)$  and present the trace of  $A$  solely by an integral of its Schwartz kernel.

**7.3. Dirac operators.** Let  $D$  be an allowable Dirac operator on  $M$ . We denote the  $\Gamma$ -equivariant lift<sup>15</sup> of  $D$  to the Galois covering  $\widetilde{M}$  by  $\widetilde{D}$ . We briefly denote by  $Q$  and  $\widetilde{Q}$  the relevant bundles of Clifford modules on which these operators act. For each  $k \in \mathbb{N}$  we can consider the edge Sobolev spaces

$$(7.4) \quad H_e^k(\widetilde{M}, \widetilde{Q}) = \{u \in L_e^2(\widetilde{M}, \widetilde{Q}) : Pu \in L_e^2(\widetilde{M}, \widetilde{Q}) \, \forall P \in \mathrm{Diff}_e^k(\widetilde{M}, \widetilde{Q})^\Gamma\}$$

where  $\mathrm{Diff}_e^*(\widetilde{M}, \widetilde{Q})^\Gamma$  is the algebra of  $\Gamma$ -equivariant edge differential operators on  $\widetilde{M}$  and where the subscript  $e$  in  $L_e^2$  refers to the complete edge metric  $\widetilde{g}_e$  associated to  $\widetilde{g}$ . Recall that the  $L^2$ -space for the incomplete edge metric  $\widetilde{g}$ ,  $L^2$ , and  $L_e^2$  are obtained one from the other from the relation  $L_e^2 = x^{\dim M/2} L^2$ . The *incomplete* edge Sobolev spaces, denoted  $H_{ie}^k(\widetilde{M}, \widetilde{Q})$  are defined similarly:  $H_{ie}^k := x^{\dim M/2} H_e^k$ .

Incomplete edge differential operators on stratified Galois coverings have been already considered in [ALMP12]. There it is proved that the signature operator in the geometric Witt case, twisted by the Mishchenko bundle associated to  $\pi : \overline{M}_\Gamma \rightarrow \overline{M}$  and to the reduced  $C^*$ -algebra  $C_r^*\Gamma$ , defines an index class in  $K_*(C_r^*\Gamma)$ , a result that should be regarded as a generalisation of the Fredholm property for  $D$  itself. The crucial remark in [ALMP12], used again in [ALMP15], is that the microlocal techniques used for  $D$  applies equally well for the Mishchenko-twisted version of  $D$  once we observe that the Galois covering over a distinguished neighbourhood  $\mathbb{R}^b \times C(F)$  is trivial, see the Proof of Proposition 6.3 in [ALMP12]. In this paper we are interested in Von Neumann index theory but the crucial remark applies equally well. Thus, using the

<sup>15</sup>Here we really need  $D$  to be a differential operator in order to obtain a well-defined lift.

same ideas as in the proof of Proposition 6.3 in [ALMP12] one can prove the following result (see also [ALMP13]).

**Proposition 7.3.** *The Dirac operator  $\tilde{D}$  satisfies the following properties.*

- (i) *Its maximal domain is given by  $\mathcal{D}_{\max}(\tilde{D}) \subset \bigcap_{\epsilon > 0} x^{1-\epsilon} H_{ie}^1(\tilde{M}, \tilde{Q})$ ;*
- (ii)  *$\mathcal{D}_{\min}(\tilde{D}) = \mathcal{D}_{\max}(\tilde{D})$  so that  $\tilde{D}$  is essentially self-adjoint<sup>16</sup>;*
- (iii) *the domain of the unique self-adjoint extension of  $(\text{Id} + \tilde{D}^2)^{-N}$ ,  $N > \dim M$ , is contained in  $\bigcap_{\epsilon > 0} x^{1-\epsilon} H_{ie}^{2N}(\tilde{M}, \tilde{Q})$*

From there we conclude with the following statement on the trace class property.

**Proposition 7.4.** *If  $N > \dim M/2$  then  $(1 + \tilde{D}^2)^{-N}$  is  $\Gamma$ -trace class. Consequently, for each rapidly decreasing function  $f$ , the operator  $f(\tilde{D})$  is  $\Gamma$ -trace class. In particular, the operators  $e^{-t\tilde{D}^2}$  and  $\tilde{D}e^{-t\tilde{D}^2}$  are  $\Gamma$ -trace class. Moreover, if  $\chi_x$  is the characteristic function of  $[-x, x] \subset \mathbb{R}$  for any  $x \geq 0$ , then  $E_x := \chi_x(\tilde{D})$  is  $\Gamma$ -trace class as well.*

*Proof.* From the third point above we deduce that  $\tilde{D}$  has the property that

$$(7.5) \quad \phi(\text{Id} + \tilde{D}^2)^{-N} \psi \text{ is trace class in } L^2$$

for each  $\phi$  and  $\psi$  of compact support. By the analogue in our context of [At76, Lemma 4.8] we infer that  $(1 + \tilde{D}^2)^{-N}$  is  $\Gamma$ -trace class. The proof that  $f(\tilde{D})$  is  $\Gamma$ -trace class now proceed as in the proof of Proposition 5.5. Finally, note that we may write  $\chi_x = f \circ \chi_x$  for an appropriately chosen rapidly decreasing function  $f$ . Thus  $E_x$  is  $\Gamma$ -trace class because  $f(\tilde{D})$  is  $\Gamma$ -trace class and  $\chi_x(\tilde{D})$  is bounded.  $\square$

As a consequence, the operators  $e^{-t\tilde{D}^2}$  and  $\tilde{D}e^{-t\tilde{D}^2}$  are  $\Gamma$ -trace class. However, we point out that the integral representation of their corresponding traces requires an additional argument which is worked out below in §7.5.

**7.4. On essential self-adjointness.** For the sake of completeness we also offer a different proof of essential self-adjointness.

**Proposition 7.5.** *The operator  $\tilde{D}$  is essentially self-adjoint in  $L^2(\tilde{M}, Q)$ .*

*Proof.* We provide for brevity only a rough idea of the proof, since the arguments involved are classical by now. First of all note that elements of  $\mathcal{D}_{\max}(\tilde{D})$  admit a weak asymptotic expansion at the edges of  $\tilde{M}$ . This can be done exactly as in Mazzeo [Maz91, Theorem 7.3], where the expansion is obtained using only Mellin transformation and applies generally to a non-compact setting as well.

We then use the fact that elements in  $\mathcal{D}_{\max}(\tilde{D})$  may be approximated in the graph norm by elements in the maximal domain that in addition are polyhomogeneous in their asymptotics at the edge. This is done precisely as in the proof of Vertman [Ver15, Proposition 3.2], cf. also Mazzeo-Vertman [MAVe12]. This in turn can be used to

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<sup>16</sup>We will offer a different proof of essential self-adjointness of  $\tilde{D}$  below in §7.4.



show that  $\mathcal{D}_{\min}(\tilde{D}) = \mathcal{D}_{\max}(\tilde{D})$  assuming the geometric Witt condition, cf. [MAVE12, Lemma 2.4] and [VER15, Proposition 3.2].  $\square$

**7.5. Heat kernel on Galois coverings and  $\Gamma$ -trace integral formula.** We define the heat operator  $e^{-t\tilde{D}^2}$  for  $\tilde{D}^2$  using functional calculus. By Proposition 7.4 we know that  $e^{-t\tilde{D}^2}$  is  $\Gamma$ -trace class. Instead of developing a microlocal heat calculus for singular Galois coverings as in [MAVE12], we employ the framework worked out by Lesch [LES94] and prove the following result.

**Proposition 7.6.** *Consider any functions  $\phi, \psi$  that are smooth on the resolution of  $M$  as a compact manifold with boundary. Lift  $\phi, \psi$  to functions on the Galois covering with compact support. Then the operators  $\phi e^{-t\tilde{D}^2}\psi$  and  $\phi(\tilde{D}e^{-t\tilde{D}^2})\psi$  are trace class and, under the isometric embedding  $\mathcal{F} \hookrightarrow M$ , the trace norms satisfy the following asymptotic behaviour for any  $N \in \mathbb{N}$  and any  $k \in \mathbb{N}$*

$$(7.6) \quad \begin{aligned} \|\phi(e^{-t\tilde{D}^2} - e^{-tD^2})\psi\|_{\text{tr}} &= O(t^N), \quad t \rightarrow 0, \\ \|\phi(\tilde{D}^k e^{-t\tilde{D}^2} - D^k e^{-tD^2})\psi\|_{\text{tr}} &= O(t^N), \quad t \rightarrow 0. \end{aligned}$$

*Proof.* We know that the fundamental domain  $\mathcal{F}$  can be embedded isometrically into the base  $M$ , with complement of measure 0. Moreover  $D$  and  $\tilde{D}$  coincide over  $M$  in the sense of [LES94, Definition 2.8]. Here, essential self-adjointness of  $D$  and  $\tilde{D}$  is crucial in order to ensure that the condition [LES94, (2.14)] holds. Recall that

$$(7.7) \quad (\text{Id} + D^2)^{-m-1} \text{ is trace class in } L^2(M, Q).$$

Thus  $D$  satisfies the *Shatten property* introduced in [LES94, Lemma 2.5]. We know that the lifted operator  $\tilde{D}$  satisfies a similar Shatten property, namely

$$\phi(\text{Id} + \tilde{D}^2)^{-m-1}\psi \text{ is trace class in } L^2(\tilde{M}, Q)$$

Now the statement follows from [LES94, Lemma 2.10].  $\square$

**Corollary 7.7.** *The Schwartz kernels of  $e^{-t\tilde{D}^2}$  and  $\tilde{D}e^{-t\tilde{D}^2}$  are smooth<sup>17</sup> in  $\tilde{M} \times \tilde{M}$ .*

*Proof.* The previous Proposition 7.6 in particular implies that for any  $\phi, \psi \in C^\infty(\mathcal{F})$  with compact support away from the edge singularities, the operator  $\phi(\tilde{D}^k e^{-t\tilde{D}^2})\psi$  is a bounded operator on  $L^2(U)$ , where  $U := \text{supp } \phi \cup \text{supp } \psi$ . Since  $k \in \mathbb{N}$  is arbitrary we conclude by elliptic regularity that  $\phi(\tilde{D}^k e^{-t\tilde{D}^2})\psi$  is smoothing and hence admits a smooth integral kernel. Consequently, the Schwartz kernels of  $e^{-t\tilde{D}^2}$  and  $\tilde{D}e^{-t\tilde{D}^2}$  are indeed smooth in the open smooth manifold  $\mathcal{F} \times \mathcal{F}$  and hence also on  $\tilde{M} \times \tilde{M}$ .  $\square$

We conclude the subsection with an integral representation of the  $\Gamma$ -trace in terms of the Schwartz kernel for the operators  $e^{-t\tilde{D}^2}$  and  $\tilde{D}e^{-t\tilde{D}^2}$ . We proceed using the argument of Proposition 5.5 and establish the following result.

<sup>17</sup>Note that we do not claim smoothness of the integral kernels if  $\tilde{M}$  is replaced with its resolution, which is a smooth manifold with boundary.

**Proposition 7.8.** *The operators  $e^{-t\tilde{D}^2}$  and  $\tilde{D}e^{-t\tilde{D}^2}$  are  $\Gamma$ -trace class and their traces can be represented by integrals of their corresponding Schwartz kernels*

$$(7.8) \quad \begin{aligned} \mathrm{Tr}_\Gamma \left( e^{-t\tilde{D}^2} \right) &= \int_{\mathcal{F}} \left( e^{-t\tilde{D}^2} \right) (p, p) \mathrm{dvol}(p), \\ \mathrm{Tr}_\Gamma \left( \tilde{D}e^{-t\tilde{D}^2} \right) &= \int_{\mathcal{F}} \left( \tilde{D}e^{-t\tilde{D}^2} \right) (p, p) \mathrm{dvol}(p). \end{aligned}$$

*Proof.* Using the semi-group property of the heat operator, we can write the operators  $e^{-t\tilde{D}^2}$  and  $\tilde{D}e^{-t\tilde{D}^2}$  as products of two  $\Gamma$ -Hilbert Schmidt operators

$$(7.9) \quad \tilde{D}e^{-t\tilde{D}^2} = (\tilde{D}e^{-\frac{t}{2}\tilde{D}^2})(e^{-\frac{t}{2}\tilde{D}^2}), \quad e^{-t\tilde{D}^2} = (e^{-\frac{t}{2}\tilde{D}^2})(e^{-\frac{t}{2}\tilde{D}^2}).$$

In view of the integral formula in (7.1) and Remark 7.2, it suffices to prove continuity of the Schwartz kernels of  $e^{-t\tilde{D}^2}$  and  $\tilde{D}e^{-t\tilde{D}^2}$  at the diagonal of  $\mathcal{F} \times \mathcal{F}$ . This follows from Corollary 7.7.  $\square$

**7.6. Eta invariants on Galois coverings with edge singularities.** Following Ramachandran [RAM93, (3.1.13)] we know that there exists a tempered measure  $m_\Gamma$  on  $\mathbb{R}$  such that  $\int_{\mathbb{R}} (1 + |x|^\ell)^{-1} dm_\Gamma$  is finite for some positive integer  $\ell \in \mathbb{N}$  and

$$\begin{aligned} \mathrm{tr}_\Gamma (\tilde{D}e^{-t\tilde{D}^2}) &= \int_{\mathcal{F}} (\tilde{D}e^{-t\tilde{D}^2})(p, p) \mathrm{dvol}(p) = \int_{\mathbb{R}} x e^{-tx^2} dm_\Gamma(x), \\ \mathrm{tr}_\Gamma (e^{-t\tilde{D}^2}) &= \int_{\mathcal{F}} (e^{-t\tilde{D}^2})(p, p) \mathrm{dvol}(p) = \int_{\mathbb{R}} e^{-tx^2} dm_\Gamma(x), \end{aligned}$$

The measure  $m_\Gamma$  is defined by the functional

$$\mathcal{S}(\mathbb{R}) \ni f \rightarrow \mathrm{tr}_\Gamma(f(\tilde{D}))$$

which we know is well defined. We can now estimate exactly as in Ramachandran [RAM93, (3.1.17)]

$$(7.10) \quad \begin{aligned} &\left| \frac{1}{\Gamma(1/2)} \int_1^\infty t^{-1/2} \mathrm{tr}_\Gamma (\tilde{D}e^{-t\tilde{D}^2}) dt \right| \\ &\leq \frac{1}{\Gamma(1/2)} \int_{\mathbb{R}} |x| e^{-x^2} \left( \int_1^\infty t^{-1/2} e^{-(t-1)x^2} dt \right) dm_\Gamma(x) \\ &\leq \int_{\mathbb{R}} e^{-x^2} dm_\Gamma(x) = \mathrm{tr}_\Gamma (e^{-\tilde{D}^2}). \end{aligned}$$

We can then associate to  $\tilde{D}$  an  $\Gamma$  eta function

$$\eta_\Gamma(\tilde{D}, s) = \frac{1}{\Gamma((s+1)/2)} \left( \int_0^1 + \int_1^\infty \right) t^{(s-1)/2} \mathrm{tr}_\Gamma (\tilde{D}e^{-t\tilde{D}^2}) dt,$$

where the integral over  $[1, \infty)$  is bounded in a neighborhood of  $s = 0$  by the estimate (7.10). The integral over  $(0, 1)$  is well-defined for  $\mathrm{Re}(s) > (m+1)/2$  and admits a meromorphic extension to  $\mathbb{C}$  by the short time asymptotics of  $\mathrm{tr}_\Gamma (\tilde{D}e^{-t\tilde{D}^2})$  which follows from Proposition 7.6 and Theorem 5.6. This yields the following a priori statement, which parallels the statement of Propositions 6.1 and 6.2.

- Proposition 7.9.** (i) *the  $\Gamma$ -eta invariant  $\eta_\Gamma(D)$  is well-defined if  $m$  is even, and  $(D, b)$  are either both of even or both of odd parity.*
- (ii) *the residue of  $\eta_\Gamma(D, s)$  at  $s = 0$  comes from the interior only, if  $m$  is odd, and  $(D, b)$  are either both of even or both of odd parity.*
- (iii) *the residue of  $\eta_\Gamma(D, s)$  at  $s = 0$  comes from the interior and from the edge fibration, if  $D$  is even and either  $(m - b)$  is odd or  $b$  is even; or if  $D$  is odd and either  $(m - b)$  is even or  $b$  is odd.*
- (iv) *the  $\Gamma$ -eta function  $\eta_\Gamma(D, s)$  may admit a second order pole singularity at  $s = 0$  if  $D$  is even,  $(m - b)$  is even and  $b$  is odd; or if  $D$  is odd,  $(m - b)$  is odd and  $b$  is even.*

In the special case of  $D$  being the Gauss-Bonnet, the signature or the spin Dirac operator, we employ Theorem 5.7 to obtain a much stronger statement.

- Proposition 7.10.** (i) *the  $\Gamma$ -eta invariant  $\eta_\Gamma(D)$  is identically zero if  $m$  is even.*
- (ii) *the  $\Gamma$ -eta invariant  $\eta_\Gamma(D, s)$  has at most a first order pole singularity at  $s = 0$ , if  $m$  is odd. Then the residue at  $s = 0$  always comes from the edge fibration only, i.e. is given by an integral over  $\partial M$  of a quantity that depends locally on the fibration  $\phi : \partial M \rightarrow B$  and the asymptotics of the Riemannian metric  $g$  at the edge.*

## 8. THE INDEX THEOREM FOR MANIFOLDS WITH EDGE SINGULAR BOUNDARY

Let  $(M, g)$  be a compact incomplete manifold with edge singularities and an admissible Riemannian metric. Assume  $M$  is boundary of another compact admissible edge manifold  $X$ . We equip  $X$  with a product metric  $(du^2 + g)$  near the boundary, where  $u \in [0, \varepsilon)$  is the inward normal coordinate. Consider the closed double manifold  $X_d = X \cup_M (-X)$ . By the product assumption, the Riemannian metric on  $X$  extends smoothly to a Riemannian metric on the closed double.

Consider an allowable Dirac-type operator  $\mathbb{D}$  acting between sections of flat Hermitian vector bundles  $E$  and  $F$  over  $X$ . Assume that over the collar  $M \times [0, \varepsilon)$  of the boundary,  $\mathbb{D}$  takes a special form

$$(8.1) \quad \mathbb{D} = \sigma \left( \frac{\partial}{\partial u} + D \right),$$

where  $u \in [0, \varepsilon)$  is the inward normal coordinate and  $\sigma$  is a bundle isomorphism  $E \upharpoonright M \rightarrow F \upharpoonright M$ . Here, the tangential operator  $D$  is a self-adjoint operator acting on sections of  $E \upharpoonright M$ . Consider its closed double  $X_d$  and extend the bundles  $(E, F)$  and the Dirac operator  $\mathbb{D}$  canonically to  $X_d$ . Assume that the Dirac operator  $\mathbb{D}$  extended to the closed double  $X_d$  satisfies the geometric Witt condition and is even or odd. We will continue under this assumption throughout the paper.

**8.1. Index theory for manifolds with edge singular boundary.** We fix a closed domain of  $\mathbb{D}$  by putting Atiyah-Patodi-Singer boundary conditions  $P_+(D)$  at  $\partial X = M$ , defined in terms of the positive spectral projection of  $D$ . We do not need to impose boundary conditions at the singular strata of  $X$  due to the geometric Witt condition. More precisely, we single out the subspace of elements in  $\mathcal{D}_{\max}(\mathbb{D})$  that are smooth up to the boundary. We denote them by  $\mathcal{D}_{\max}^\infty(\mathbb{D})$ . This defines the core domain

$$(8.2) \quad \mathcal{D}_+^c(\mathbb{D}) := \{u \in \mathcal{D}_{\max}^\infty(\mathbb{D}) \mid (P_+(D)u)|_{\partial X} = 0\}.$$

We fix the closed extension of  $\mathbb{D}$  with domain  $\mathcal{D}(\mathbb{D})$  defined as the graph-closure of the core domain  $\mathcal{D}_+^c(\mathbb{D})$  in  $L^2(X, E^* \oplus F)$ .

**Proposition 8.1.**  *$\mathbb{D}$  is Fredholm with index*

$$\text{index } \mathbb{D} = \text{Tr } e^{-t\mathbb{D}^*\mathbb{D}} - \text{Tr } e^{-t\mathbb{D}\mathbb{D}^*}.$$

*Proof.* Consider first the model situation with  $X_0 := M \times [0, \infty)$ . We define a Dirac operator  $\mathbb{D}_0$  on  $X_0$  by (8.1). As in case of  $\mathbb{D}$ , we fix a closed domain for  $\mathbb{D}_0$  by putting Atiyah-Patodi-Singer boundary conditions  $P_+(D)$  at  $\partial X_0 = M$ . Since  $D$  is essentially self-adjoint with discrete spectrum, the heat kernels  $H_1$  and  $H_2$  for  $\mathbb{D}_0^*\mathbb{D}_0$  and  $\mathbb{D}_0\mathbb{D}_0^*$ , respectively, can be constructed explicitly using spectral eigenspace decomposition on  $M$ , as in [APS75A, p. 10-11]. Writing  $(s, p), (s', p') \in M \times [0, \infty)$  for two copies of coordinates on  $X_0$ , we infer from [APS75A, (2.20)] that the heat kernels  $H_j(t, s, p, s', p'), j = 1, 2$ , are bounded by

$$(8.3) \quad \frac{3}{2\sqrt{\pi t}} \left( \text{tr } e^{-\frac{t}{2}D^2}(p, p) + \text{tr } e^{-\frac{t}{2}D^2}(p', p') \right) e^{-\frac{(s-s')^2}{4t}}.$$

From here one easily concludes that for any smooth cutoff functions  $\phi, \psi \in C^\infty(X_0)$  with compact support the kernels  $\phi H_j \psi, j = 1, 2$ , are Hilbert Schmidt in  $L^2(X_0)$ .

The heat kernels  $H_1, H_2$  solve the heat equation for  $\mathbb{D}^*\mathbb{D}$  and  $\mathbb{D}\mathbb{D}^*$  near the boundary  $\partial X$ , respectively, and can be corrected to an exact solution using the heat kernels from the interior of  $X$  in the usual way. The heat kernel parametrices in the interior of  $X$  may be constructed microlocally in the same way as in §3, and are Hilbert Schmidt for  $t > 0$  when multiplied from both sides with smooth cutoff functions that are compactly supported in the interior of  $X$ .

Then the heat operators for  $\mathbb{D}^*\mathbb{D}$  and  $\mathbb{D}\mathbb{D}^*$  must be Hilbert Schmidt in  $L^2(X)$  for  $t > 0$ . Consequently the heat operators for  $\mathbb{D}^*\mathbb{D}$  and  $\mathbb{D}\mathbb{D}^*$  are trace class by the semigroup property of the heat kernel. Thus both operators have discrete spectrum with finite multiplicities by the spectral theorem for compact operators. Hence  $\mathbb{D}$  and  $\mathbb{D}^*$  have finite dimensional kernels and  $\mathbb{D}$  is Fredholm. The formula for index now follows.  $\square$

**8.2. Eta invariants on singular boundaries of edge spaces.** Now we employ an argument by Atiyah-Patodi-Singer to prove well-definement of the eta invariant of the tangential operator  $D$  appearing in the formula (8.1), under certain additional assumptions.

**Theorem 8.2.** *Assume that for the dimension  $b_i$  of each edge in  $X$ , either  $(m+1-b_i)$  or  $b_i$  is odd. Then the eta-invariant  $\eta(D)$  of the tangential operator  $D$  in (8.1) is well-defined.*

*Proof.* We follow the argument of Atiyah-Patodi-Singer, which we also employ below in the proof of Theorem 8.4. We still write  $\mathbb{D}$  for the Dirac operator on the closed double. Evenness or oddness of  $\mathbb{D}$  guarantees that the heat kernels  $\exp(-t\mathbb{D}^*\mathbb{D})$  and  $\exp(-t\mathbb{D}\mathbb{D}^*)$  both lie in the even subcalculus. In [APS75A, (3.4)] the authors define a function

$$(8.4) \quad F(q, t) := \text{tr}(\exp(-t\mathbb{D}^*\mathbb{D}) - \exp(-t\mathbb{D}\mathbb{D}^*))(q, q)$$

on the double manifold  $q \in X_d$ . The short time asymptotic expansion of  $F(q, t)$  follows from (5.12) and in particular obtain ( $b_i$  denote the dimensions of the various edge singular strata in the double manifold  $X_d$ )

$$(8.5) \quad F(t) := \int_X F(q, t) \sim_{t \rightarrow 0} \sum_{\ell=0}^{\infty} A'_\ell t^{\ell - \frac{m+1}{2}} + \sum_i \sum_{\ell=0}^{\infty} B'_\ell t^{\ell - \frac{b_i}{2}} + \sum_i \sum_{\ell \in \mathfrak{J}_i} C'_\ell t^{\ell - \frac{b_i}{2}} \log t.$$

where  $\mathfrak{J}_i = \emptyset$  if  $(m+1-b_i)$  is odd and  $\mathfrak{J}_i = \mathbb{N}_0$  if  $(m+1-b_i)$  is even. Using the notation introduced in the proof of Proposition 8.1, we also define according to [APS75A, (2.23)]

$$(8.6) \quad K(t) := \int_0^\infty \int_M \text{tr}(\exp(-t\mathbb{D}_0^*\mathbb{D}_0) - \exp(-t\mathbb{D}_0\mathbb{D}_0^*))(s, p, s, p) ds \, \text{dvol}_g(p).$$

Duhamel principle implies in view of Proposition 8.1 that for any  $N \in \mathbb{N}$

$$(8.7) \quad \text{index } \mathbb{D} = \text{Tr } e^{-t\mathbb{D}^*\mathbb{D}} - \text{Tr } e^{-t\mathbb{D}\mathbb{D}^*} = K(t) + F(t) + O(t), \quad t \rightarrow 0.$$

In this context, the central observation of Atiyah-Patodi-Singer in [APS75A, (2.25)] is that

$$(8.8) \quad \int_0^\infty \left( K(t) + \frac{1}{2} \dim \ker D \right) t^{s-1} dt = -\frac{\Gamma(s + \frac{1}{2})}{2s\sqrt{\pi}} \eta(D, 2s).$$

This relation implies that the residue of  $\eta(D, 2s)$  at  $s = 0$  is given by twice the coefficient of  $\log(t)$  in the short time asymptotics of  $K(t)$ . The short time asymptotics of  $K(t)$  follows from (8.5) by the relation (8.7). Consequently, we conclude

$$(8.9) \quad \text{Res}_{s=0} \eta(D, 2s) = -2 \sum_i C'_{\frac{b_i}{2}}.$$

Assumptions on the dimensions  $b_i$  imply that the coefficients  $C'_{\frac{b_i}{2}}$  are always trivial and hence the eta invariant  $\eta(D)$  is well-defined.  $\square$

We conclude with a remark on the singularity structures allowed in  $X$ . In fact, we may allow  $X$  to admit a certain class of iterated higher order singularities. More precisely,  $X$  may admit an isolated conical singularities with an exact conical metric and the cross section being a simple edge space, or more generally be a stratified space with an essentially self-adjoint Dirac operator  $D$  with discrete spectrum as assumed in §5.4. Then, the heat kernel near such a singularity is given explicitly in terms of Bessel functions, as explained in §5.4. Then (5.12) and in particular (8.5) still holds without any  $C'_*$  coefficients coming from the conical singularity. Hence the proof of Theorem 8.2

above carries over to this slightly extended setting, if the Dirac operator  $\mathbb{D}$  is geometric, i.e. if  $\mathbb{D}^*\mathbb{D}$  and  $\mathbb{D}\mathbb{D}^*$  are the direct sum components of the Hodge de Rham or the spin Laplacians. Note that general allowable Dirac operators cannot be included, since the corresponding heat kernels cannot be described explicitly in terms of Bessel functions.

Note that such a space  $X$  with an isolated conical singularity and boundary given by  $M$  always exists by setting  $X = \mathcal{C}(M)$  where the cone extends from  $\{0\}$  (the tip) to 1, the boundary. Thus  $X^{\text{reg}} = (0, 1] \times M$  with the Riemannian metric given by  $(du^2 + u^2g)$  over  $(0, \varepsilon) \times M$  and by  $(du^2 + g)$  over  $(1 - \varepsilon, 1] \times M$  for any  $\varepsilon > 0$  small. The boundary of  $X$  is precisely  $M$ . If the Dirac operator  $D$  is equal to the Gauss-Bonnet, the signature or the spin operator, then there exists a geometric  $\mathbb{D}$  on  $X$ . We have to argue in what way the geometric Witt condition of  $D$  on  $M$  actually implies the geometric Witt condition for  $\mathbb{D}$  on  $X$ .

If  $M$  is odd dimensional and satisfies the (cohomological) Witt condition, then so does  $X$ . Assume that  $M$  actually satisfies the geometric Witt condition. Then the geometric Witt condition is satisfied also on the cone  $X$  by scaling the metric on the cross section accordingly. In the setting of the spin Dirac operator  $D$ , a different argument is needed in order to ensure the geometric Witt condition on  $X$ . There, by Lichnerowicz formula, a non-negative (eventually singular as a function of the radial function) scalar curvature on  $M$  ensures that the kernel of  $D$  is trivial. If the scalar curvature on  $M$  is positive and bounded away from zero by a sufficiently large positive constant, then the geometric Witt condition is satisfied on  $X$  as well. Assuming that the scalar curvature on  $M$  is positive and bounded away from zero, this can always be assumed by an appropriate rescaling of  $g$ . Note that scaling of  $g$  changes the eta invariant of  $D$  only by a constant factor. Consequently, we obtain the following corollary by using such specific  $X$ .

**Corollary 8.3.** *Assume that  $M$  is odd dimensional with each edge in  $M$  of even dimension. Let  $D$  be the Gauss-Bonnet operator or the signature operator, satisfying the geometric Witt condition. Then the eta-invariant  $\eta(D)$  is well-defined. If  $D$  is the spin Dirac operator satisfying the geometric Witt condition and the scalar curvature on  $M$  is positive and bounded away from zero, then the eta-invariant  $\eta(D)$  is well-defined.*

*Proof.* In view of the preceding argument, it remains to check the dimensional restrictions in the neighborhood of boundary. Note that an edge in  $M$  of dimension  $b$  gives rise to an edge in the cone  $X = (0, 1] \times M$  of dimension  $b + 1$ . Consequently, if  $b$  is even,  $(m - b)$  is odd and  $b_i$  is even, and hence  $(m + 1 - (b + 1))$  is odd and  $(b + 1)$  is odd. The dimensional conditions in Theorem 8.2 are satisfied.  $\square$

**8.3. Index formula for manifolds with edge singular boundary.** Recall, we consider an allowable Dirac-type operator  $\mathbb{D}$  acting between sections of flat Hermitian vector bundles  $E$  and  $F$  over  $X$ . We assume that over the collar  $M \times [0, \varepsilon)$  of the boundary,  $\mathbb{D}$  takes a special form (8.1). We have proved above that  $\mathbb{D}$ , equipped with APS boundary conditions is Fredholm and also established well-definiteness of the eta-invariant for its tangential operator  $D$ . We now can prove the following index formula.

**Theorem 8.4.** *Assume that  $\mathbb{D}^2$  is even, so that the heat kernel of  $\mathbb{D}^2$  on  $X_d$  lies in the even subcalculus. Assume further that for the dimension  $b_i$  of each edge singularity in*



$X_d$  either  $(m + 1 - b_i)$  or  $b_i$  is odd. Then the Fredholm index of the Dirac operator  $\mathbb{D}$  on  $X$  with APS boundary conditions is given by the index formula

$$(8.10) \quad \text{index } \mathbb{D} = \left( \int_X a_0 + \int_Y b_0 \right) - \frac{\dim \ker D + \eta(D)}{2},$$

where  $Y$  denotes the fibration of links over the edge singularities of  $X$ , the sum of the two integrals is the constant term in the short time asymptotic expansion of the trace of  $\exp(-t\mathbb{D}^*\mathbb{D}) - \exp(-t\mathbb{D}\mathbb{D}^*)$ . Moreover we have the following characterization of the integrands  $a_0$  and  $b_0$ .

- (i) The integrand  $a_0$  comes from the interior in the sense that at each  $p \in M$  the value  $a_0(p)$  depends only on the jets of the Riemannian metric and the coefficients of  $\mathbb{D}$ <sup>18</sup> at  $p$ .
- (ii) The integrand  $b_0$  comes from the edge singularity  $Y$  in  $X$  in the sense that at each  $y \in Y$  the value  $b_0(y)$  depends only on the jets of the Riemannian metric and the curvature of the link fibration over  $Y$ , which define the coefficients of  $\mathbb{D}$ <sup>19</sup> at  $(0, y, z)$ .

*Proof.* Theorem 8.2 asserts existence of the eta-invariant  $\eta(D)$ . The relation [APS75A, (3.8)] then implies that

$$(8.11) \quad \text{index } \mathbb{D} + \frac{\dim \ker D + \eta(D)}{2}$$

equals the constant term in the short time asymptotics (8.5). The statement now follows from a geometric characterization of coefficients similar to Theorem 5.8 with

$$(8.12) \quad \int_M a_0 := A'_{\frac{m+1}{2}}, \quad \int_Y b_0 := \sum_i B'_{\frac{b_i}{2}}.$$

□

This result provides a singular analogue to the index formula by Atiyah, Patodi and Singer in [APS75A, Theorem 3.10]. The integrand  $a_0$  is the same as in the classical case. In particular, for  $\mathbb{D}$  being the spin Dirac operator acting between the sections of positive and negative spinor bundles, then  $a_0$  is given by the  $\hat{A}$ -polynomial applied to the curvature tensor of  $X$ .

Atiyah, Patodi and Singer then continue and apply this formula to the various geometric settings including the signature operator in [APS75A, Theorem 4.14]. In these geometric settings, existence of a bounding manifold  $X$  is in fact not a restriction, since as in the explanation up to Corollary 8.3, we can define  $X$  by taking a finite cone over  $M$ . Theorem 8.4 holds in this case for  $M$  being odd-dimensional and edges in  $M$  being even-dimensional, as restricted in Corollary 8.3.

<sup>18</sup>where we fix any local coordinates around  $p \in M$ .

<sup>19</sup>where we fix the local coordinates  $(x, y, z)$  near the edge.

### 9. INDEX THEOREM FOR GALOIS COVERINGS WITH EDGE SINGULAR BOUNDARY

As in the previous section we consider a compact incomplete manifold  $(M, g)$  with edge singularities and an admissible Riemannian metric. Consider an essentially self-adjoint Dirac operator  $D$ , satisfying the geometric Witt condition. Assume  $M$  is boundary of another compact admissible edge manifold  $X$  with a product type metric at the boundary. Consider a symmetric Dirac-type  $\mathbb{D}$  acting between sections of a Hermitian vector bundle  $E$  over  $X$ , which is assumed to be either the spin Dirac or an allowable Dirac operator acting on twisted differential forms. Assume that over the collar  $M \times [0, \varepsilon)$  of the boundary,  $\mathbb{D}$  takes a special form

$$(9.1) \quad \mathbb{D} = \sigma \left( \frac{\partial}{\partial u} + D \right),$$

where  $u \in [0, \varepsilon)$  is the inward normal coordinate and  $\sigma$  is a bundle isomorphism on  $E|_M$ . Impose Atiyah-Patodi-Singer boundary conditions at  $M$ , defined in terms of the positive spectral projection of  $D$ . The corresponding closed extension of  $\mathbb{D}$  is defined by specifying its core domain as in (8.2).

Consider a Galois covering  $\tilde{X}$  of  $X$  with Galois group  $\Gamma$ . We denote its fundamental domain by  $\mathcal{F}$ . Its boundary is again a Galois covering  $\tilde{M}$  of  $M$ . We denote the fibration of links over the edge singular stratum in  $\tilde{X}$  by  $\tilde{Y}$ , which is again a Galois covering of the corresponding fibration in  $X$  with fundamental domain  $\mathcal{F}_Y$ .

We denote the lifts of  $\mathbb{D}$ ,  $D$  and  $\sigma$  to the coverings  $\tilde{X}$  and  $\tilde{M}$  by  $\tilde{\mathbb{D}}$ ,  $\tilde{D}$  and  $\tilde{\sigma}$ , respectively. Near the boundary  $\partial\tilde{X} = \tilde{M}$ , with the boundary defining function  $\tilde{u}$  being the lift of  $u$ , the operator  $\tilde{\mathbb{D}}$  takes the special form

$$(9.2) \quad \tilde{\mathbb{D}} = \tilde{\sigma} \left( \frac{\partial}{\partial \tilde{u}} + \tilde{D} \right).$$

**9.1. Index theory for Galois coverings with edge singular boundary.** We may now proceed to establish  $\Gamma$ -Fredholmness of  $\tilde{\mathbb{D}}$  by modelling the argument of Roe [ROE99] and Ramachandran [RAM93] in a singular setting. A related argument is given by Vaillant [VAI08], however rather with non-compact bases with cylindrical ends and hence with an  $L^2$ -version of an index theorem on Galois coverings. However, in our setting a more straightforward argument is appropriate.

First of all we note that the concepts of  $\Gamma$ -Hilbert Schmidt and  $\Gamma$ -trace class operators introduced in §7.2 carry over verbatim to the setting of Galois coverings with regular boundary. We may now state the following central proposition.

**Proposition 9.1.** *The heat operators of  $\tilde{\Delta}_1 := \tilde{\mathbb{D}}^* \tilde{\mathbb{D}}$  and  $\tilde{\Delta}_2 := \tilde{\mathbb{D}} \tilde{\mathbb{D}}^*$ , as well as the orthogonal projections  $P_{\ker \tilde{\mathbb{D}}}$  and  $P_{\ker \tilde{\mathbb{D}}^*}$  of  $L^2(\tilde{X})$  onto the kernel of  $\tilde{\mathbb{D}}$  and  $\tilde{\mathbb{D}}^*$  respectively, are  $\Gamma$ -trace class. The  $\Gamma$ -equivariant Dirac operator  $\tilde{\mathbb{D}}$  is  $\Gamma$ -Fredholm, i.e. admits a finite  $\Gamma$ -index*

$$(9.3) \quad \text{index}_\Gamma \tilde{\mathbb{D}} := \text{Tr}_\Gamma(P_{\ker \tilde{\mathbb{D}}}) - \text{Tr}_\Gamma(P_{\ker \tilde{\mathbb{D}}^*}).$$

*Proof.* We may argue as in Proposition 8.1 that the heat operators of  $\tilde{\Delta}_1$  and  $\tilde{\Delta}_2$  are  $\Gamma$ -trace class. Using the Duhamel principle it suffices to study the heat kernels of the corresponding operators on the infinite half-cylinder  $\tilde{M} \times [0, \infty)$  and the double Galois covering  $\tilde{X}_d := \tilde{X} \cup_{\tilde{M}} (-\tilde{X})$ . The latter space is smooth across the join, since the Riemannian metric on  $\tilde{X}$  is assumed to be product near the boundary  $\tilde{M} \times \{u = 0\}$ .

Arguing verbatim to Theorem 7.6, we find that the heat operators on the double Galois covering  $\tilde{X}_d := \tilde{X} \cup_{\tilde{M}} (-\tilde{X})$  are  $\Gamma$ -trace class for each fixed  $t > 0$  with a short time asymptotics as  $t \rightarrow 0$  of the same structure as in the compact setting, cf. (5.12).

By Theorem 7.6 the heat operator of  $\tilde{D}^2$  on  $\tilde{M}$  is  $\Gamma$ -trace class, and consequently the heat operators of  $\tilde{\Delta}_1$  and  $\tilde{\Delta}_2$ , viewed as operators on the infinite half-cylinder  $\tilde{M} \times [0, \infty)$ , are  $\Gamma$ -trace class near the boundary. Indeed, consider cutoff functions  $\phi, \psi \in C_0^\infty[0, \varepsilon)$  such that  $\text{supp } \phi \subset \text{supp } \psi$  and  $\text{supp } \phi \cap \text{supp } (d\psi/du) = \emptyset$ . Extend  $\phi, \psi$  trivially to functions on  $X$  and let  $\tilde{\phi}, \tilde{\psi}$  denote the lifts of  $\phi, \psi$  to the Galois covering  $\tilde{X}$ , respectively. Then, like in (8.3), compare also [RAM93, p.344-345], we may estimate for any  $k \in \{1, 2\}$  the  $\Gamma$ -trace of  $\tilde{\phi} \circ e^{-t\tilde{\Delta}_k} \circ \tilde{\psi}$  by the  $\Gamma$ -trace of  $\tilde{D}^2$  on  $\tilde{M}$ .

Consequently  $e^{-t\tilde{\Delta}_k} \in \mathcal{C}_1$  for any  $k \in \{1, 2\}$ . We can write the projections  $P_{\ker \tilde{\mathbb{D}}}$  and  $P_{\ker \tilde{\mathbb{D}}^*}$  as compositions with the corresponding heat operators

$$(9.4) \quad P_{\ker \tilde{\mathbb{D}}} = e^{-t\tilde{\Delta}_1} \circ P_{\ker \tilde{\mathbb{D}}}, \quad P_{\ker \tilde{\mathbb{D}}^*} = e^{-t\tilde{\Delta}_2} \circ P_{\ker \tilde{\mathbb{D}}^*}.$$

Since the trace class operators  $\mathcal{C}_1$  form a two-sided ideal in the von Neumann algebra  $\mathcal{A}_\Gamma(\tilde{X})$ , we conclude that the projections are  $\Gamma$ -trace class as well. Consequently the  $\Gamma$ -dimensions  $\dim_\Gamma \ker \tilde{\mathbb{D}} = \text{Tr}(P_{\ker \tilde{\mathbb{D}}})$  and  $\dim_\Gamma \ker \tilde{\mathbb{D}}^* = \text{Tr}(P_{\ker \tilde{\mathbb{D}}^*})$  are finite and  $\tilde{\mathbb{D}}$  is  $\Gamma$ -Fredholm with index

$$(9.5) \quad \text{index}_\Gamma \tilde{\mathbb{D}} := \dim_\Gamma \ker \tilde{\mathbb{D}} - \dim_\Gamma \ker \tilde{\mathbb{D}}^*.$$

□

Repeating [ROE99, Lemma 15.11], we deduce the McKean Singer index formula.

**Proposition 9.2.** *The Dirac operator  $\tilde{\mathbb{D}}$  is  $\Gamma$ -Fredholm with the index  $\text{ind}_\Gamma \tilde{\mathbb{D}}$  given by a  $\Gamma$ -super trace*

$$\text{ind}_\Gamma \tilde{\mathbb{D}} = \text{Str}_\Gamma \left( e^{-t\tilde{\mathbb{D}}^2} \right) := \text{Tr}_\Gamma \left( e^{-t(\tilde{\mathbb{D}})^*\tilde{\mathbb{D}}} \right) - \text{Tr}_\Gamma \left( e^{-t\tilde{\mathbb{D}}\tilde{\mathbb{D}}^*} \right)$$

**9.2. Eta invariants on boundaries of singular Galois coverings.** Following the Atiyah-Patodi-Singer argument outlined in the proof of Theorem 8.2, we obtain the following corresponding existence theorem on Galois coverings.

**Theorem 9.3.** *Assume the Galois covering  $\tilde{M} \rightarrow M$  is boundary of some Galois covering  $\tilde{X} \rightarrow X$  with  $\partial X = M$  and product metric and bundle structures near the boundary. Assume that for the dimension  $b_i$  of each edge in  $X$ , either  $(m+1-b_i)$  or  $b_i$  is odd. We finally assume that  $\mathbb{D}$  is an allowable even or odd Dirac operator, satisfying the geometric Witt condition. Then the  $\Gamma$  eta-invariant  $\eta_\Gamma(\tilde{D})$  of the Dirac operator  $\tilde{D}$  on  $\tilde{M}$  is well-defined.*

*Proof.* We follow the argument of Atiyah-Patodi-Singer, cf. proof of Theorem 8.2. We equip the Galois covering space  $\tilde{X}$  with a product metric  $(du^2 + g)$  near the boundary, where  $u \in [0, \varepsilon)$  is the inward normal coordinate. Consider the double manifold  $\tilde{X}_d := \tilde{X} \cup_{\tilde{M}} (-\tilde{X})$ . By the product assumption, the Riemannian metric on  $X$  extends smoothly to a Riemannian metric on the double. The Dirac  $\tilde{\mathbb{D}}$  defines an operator on the double, which we denote as  $\tilde{\mathbb{D}}$  again.

Let us now consider the heat operators associated to  $\tilde{\mathbb{D}}$  as a Dirac operator on the double manifold  $\tilde{X}_d$ . By the same argument as in Corollary 7.7, we conclude continuity of the Schwartz kernels for  $\exp(-t\tilde{\mathbb{D}}^*\tilde{\mathbb{D}})$  and  $\exp(-t\tilde{\mathbb{D}}\tilde{\mathbb{D}}^*)$  at the diagonal and hence a representation of their traces by integrals of their respective Schwartz kernels at the diagonal along the fundamental domain, as in Proposition 7.8.

Evenness or oddness of  $\tilde{\mathbb{D}}$  guarantees that the heat kernels  $\exp(-t\tilde{\mathbb{D}}^*\tilde{\mathbb{D}})$  and  $\exp(-t\tilde{\mathbb{D}}\tilde{\mathbb{D}}^*)$  both lie in the even subcalculus. In [APS75A, (3.4)] the authors define a function

$$(9.6) \quad F(p, t) := \text{tr} \left( \exp(-t\tilde{\mathbb{D}}^*\tilde{\mathbb{D}}) - \exp(-t\tilde{\mathbb{D}}\tilde{\mathbb{D}}^*) \right) (p, p)$$

on the double manifold  $\tilde{X}_d$ . The short time asymptotic expansion of  $F(p, t)$  follows from Theorem 7.6 and (5.12) and in particular obtain ( $b_i$  denote the dimensions of the various edge singular strata in the double manifold  $\tilde{X}_d$ )

$$(9.7) \quad F(t) := \int_{\mathcal{F}} F(p, t) \sim_{t \rightarrow 0} \sum_{\ell=0}^{\infty} A'_\ell t^{\ell - \frac{m+1}{2}} + \sum_i \sum_{\ell=0}^{\infty} B'_\ell t^{\ell - \frac{b_i}{2}} + \sum_i \sum_{\ell \in \mathfrak{I}_i} C'_\ell t^{\ell - \frac{b_i}{2}} \log t.$$

where  $\mathfrak{I}_i = \emptyset$  if  $(m+1-b_i)$  is odd and  $\mathfrak{I}_i = \mathbb{N}_0$  if  $(m+1-b_i)$  is even. In the present context, the central observation of Atiyah-Patodi-Singer in [APS75A, (3.8)], worked out in detail in the proof of Proposition 8.2 and carries over verbatim to the non-compact setting of Galois coverings, is that

$$(9.8) \quad \text{Res}_{s=0} \eta_\Gamma(\tilde{D}, 2s) = -2 \sum_i C'_{\frac{b_i}{2}}.$$

Assumptions on the dimensions  $b_i$  imply that the coefficients  $C'_{\frac{b_i}{2}}$  are always trivial and hence the eta invariant  $\eta_\Gamma(\tilde{D})$  is well-defined.  $\square$

**9.3. Atiyah-Patodi-Singer index theorem on Galois coverings.** We may now proceed as in [RAM93, Theorem 7.1.1] to deduce an index formula for the  $\Gamma$ -index of  $\tilde{\mathbb{D}}$ , where the index problem on the Galois covering is reduced to the base, where we apply the same argument as in Theorem 8.4 to deduce the following

**Theorem 9.4.** *Assume that  $\tilde{X}$  is an admissible edge space with dimension  $b_i$  of each edge singularity being such that either  $(m+1-b_i)$  or  $b_i$  is odd. Assume  $\tilde{\mathbb{D}}^2$  is even, so that the heat kernel of  $\tilde{\mathbb{D}}^2$  on the double Galois covering  $\tilde{X}_d$  lies in the even subcalculus.*

Then the  $\eta_\Gamma(\tilde{D})$  eta invariant is well-defined by restricting the eta function

$$\eta_\Gamma(\tilde{D}, s) = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2} \operatorname{tr}_\Gamma(\tilde{D}e^{-t\tilde{D}^2}) dt$$

to  $s = 0$  and is related to the  $\Gamma$ -Fredholm index of  $\tilde{\mathbb{D}}$  by the index formula

$$(9.9) \quad \operatorname{ind}_\Gamma \tilde{\mathbb{D}} = \left( \int_{\mathcal{F}} \tilde{a}_0 + \int_{\mathcal{F}_Y} \tilde{b}_0 \right) - \frac{\dim \ker_\Gamma \tilde{D} + \eta_\Gamma(\tilde{D})}{2},$$

where the sum of two integrals is the constant term in the short time asymptotic expansion of the super trace  $\operatorname{Str}_\Gamma(e^{-t\tilde{\mathbb{D}}^2})$  on the double Galois covering  $\tilde{X} \cup_{\tilde{M}} (-\tilde{X})$ .

Moreover we have the following characterization of the integrands  $\tilde{a}_0$  and  $\tilde{b}_0$ .

- (i) The integrand  $\tilde{a}_0$  comes from the interior in the sense that at each  $p \in \mathcal{F}$  the value  $\tilde{a}_0(p)$  depends only on the jets of the Riemannian metric and the coefficients of  $\tilde{\mathbb{D}}^{20}$  at  $p$ .
- (ii) The integrand  $\tilde{b}_0$  comes from the edge singularity  $\tilde{Y}$  in  $\tilde{X}$  in the sense that at each  $y \in \mathcal{F}_Y$  the value  $\tilde{b}_0(y)$  depends only on the jets of the Riemannian metric and the coefficients of  $\tilde{\mathbb{D}}^{21}$  at  $(0, y, z)$ .

## 10. EXISTENCE OF THE APS AND CHEEGER-GROMOV RHO INVARIANTS

**10.1. Existence of the Atiyah-Patodi-Singer rho invariants.** Let us first consider the Atiyah-Patodi-Singer rho invariant on an edge singular manifold  $(M, g)$  with an admissible edge metric. Let  $E$  and  $F$  denote two flat vector bundles induced by two unitary representations  $\alpha$  and  $\beta$  of the fundamental group  $\pi_1(\overline{M})$ , respectively, where  $\overline{M}$  denotes the compact stratified space including its singular stratum. Each vector bundle yields a twisted Dirac operator, which we denote by  $D_E$  and  $D_F$ , respectively. Assume both satisfy the geometric Witt condition. Consider the difference of the associated eta functions.

$$(10.1) \quad \rho_{\alpha-\beta}(s, D) = \eta(s, D_E) - \eta(s, D_F).$$

In case, each individual eta function is regular at  $s = 0$ , the rho function is regular at  $s = 0$  and its value at zero, the Atiyah-Patodi-Singer (APS) rho invariant, obviously exists. Otherwise the residues of both eta functions cancel by Theorem 5.8 the rho function is still regular at  $s = 0$ .

In case the individual eta invariants are not well-defined, we may be in the setup where the residues of the eta functions at  $s = 0$  are induced by the local (interior) coefficient  $A_{\frac{m-1}{2}}$  in the asymptotic expansions in Theorem 5.6. We indicate the a priori dependence of the coefficients on the vector bundles by an additional upper index. By flatness of the vector bundles  $E$  and  $F$ , these local (!) coefficients  $A_{\frac{m-1}{2}}^E$  and  $A_{\frac{m-1}{2}}^F$  coincide and cancel in their contribution to the residue of  $\rho_{(E,F)}(s, D)$  at  $s = 0$ . We conclude

<sup>20</sup>where we fix any local coordinates around  $p \in \mathcal{F}$ .

<sup>21</sup>where we fix the local coordinates  $(x, y, z)$  near the edge.

**Theorem 10.1.** *Assume  $(M^m, g)$  is an edge manifold with an admissible edge metric and an edge singularity at  $B^b$ . Let  $D$  be an allowable Dirac operator. Let  $\alpha$  and  $\beta$  be two unitary representations of the fundamental group  $\pi_1(\overline{M})$ . Assume that both twisted Dirac operators satisfy the geometric Witt condition. Then the rho function  $\rho_{\alpha-\beta}(s, D)$  is well-defined at  $s = 0$ . The corresponding APS rho invariant is then defined by*

$$\rho_{\alpha-\beta}(D) := \rho_{\alpha-\beta}(0, D).$$

A remark is in order. In case of  $D$  being the signature or the Gauss Bonnet operator, geometric Witt condition for  $D$  does not in general imply that the twisted operators satisfy the geometric Witt condition as well. However, the cohomological Witt condition, which holds irrespective of the twisting, can always be strengthened to its geometric version via scaling of the metric on the links of the edge fibration. In case  $D$  is the spin Dirac operator, assumption of everywhere positive scalar curvature together with appropriate scaling of the metric implies by the Lichnerowicz formula that the geometric Witt condition holds for all twisted spin Dirac operators as well.

**10.2. Existence of the Cheeger-Gromov rho invariants.** Let us now consider the Cheeger-Gromov rho invariant on an admissible edge manifold  $(M, g)$  and its Galois covering  $\widetilde{M}$  with Galois group  $\Gamma$  and fundamental domain  $\mathcal{F}$ . We continue in the notation of §9.2 and consider the difference of the associated eta functions.

$$(10.2) \quad \rho_\Gamma(D, s) := \eta_\Gamma(\widetilde{D}, s) - \eta(D, s).$$

By Proposition 7.6 the short time asymptotics of the trace for  $e^{-tD^2}$  and the  $\Gamma$ -trace for  $e^{-t\widetilde{D}^2}$  coincide. Consequently we obtain the following result.

**Theorem 10.2.** *Assume  $(M^m, g)$  is an edge manifold with an admissible edge metric and an edge singularity at  $B^b$ . Let  $D$  be an allowable Dirac operator satisfying the geometric Witt condition. Let  $\widetilde{D}$  be its lift to the Galois covering  $\widetilde{M}$ . Then the rho function  $\rho_\Gamma(D, s)$  is well-defined at  $s = 0$  and the corresponding Cheeger-Gromov rho invariant is defined by*

$$\rho_\Gamma(D) := \rho_\Gamma(D, 0)$$

## 11. STABILITY PROPERTIES OF RHO-INVARIANTS

### 11.1. Rho invariants of bordant metrics of positive scalar curvature.

**Definition 11.1.** Let  $(M_1, g_1)$  and  $(M_2, g_2)$  be two incomplete edge spaces with admissible edge metrics  $g_1$  and  $g_2$  of *positive* scalar curvature. We assume that both  $(M_1, g_1)$  and  $(M_2, g_2)$  are spin. We also assume the existence of two classifying maps  $r_1 : \overline{M}_1 \rightarrow B\Gamma$  and  $r_2 : \overline{M}_2 \rightarrow B\Gamma$ . We call the two triples  $(M_1, r_1, g_1)$  and  $(M_2, r_2, g_2)$  *bordant* if there exists a spin edge space  $\overline{W}$  with boundary  $\partial\overline{W} = \overline{M}_1 \sqcup (-\overline{M}_2)$  equipped on its regular part with a positive scalar curvature metric  $G$  which is product near the boundary and restricts to  $g_1$  at the boundary component  $M_1$ , and to  $g_2$  at the boundary component  $(-\overline{M}_2)$ . We also assume the existence of a classifying map  $R : \overline{W} \rightarrow B\Gamma$  restricting to  $r_1$  and  $r_2$  at the boundary. We restrict the class of bordisms  $W$  to those with edge dimensions  $b_i$  such that either  $(m + 1 - b_i)$  or  $b_i$  is odd.



Consider Theorem 8.4 in the special case of  $D$  being the spin Dirac operator. We indicate the two spin Dirac operators associated to  $(M_1, g_1)$  and  $(M_2, g_2)$  by the corresponding lower index 1 and 2. Pulling back the universal bundle over  $B\Gamma$  with the two classifying maps we obtain two Galois coverings with base respectively  $M_1$  and  $M_2$ . We consider two unitary representations  $\alpha$  and  $\beta$  of  $\Gamma$  of the same dimension. We twist the operators with the corresponding flat vector bundles  $E_\alpha$  and  $E_\beta$  and indicate dependence of the operators on the representations by an upper script  $\alpha$  and  $\beta$ . By the Weitzenböck formula and the positive scalar curvature assumption, the index of  $\mathbb{D}$  is zero and we obtain from 8.4

$$(11.1) \quad \begin{aligned} \eta(D_1^\alpha) - \eta(D_2^\alpha) &= 2 \left( \int_X a_0^\alpha + \int_Y b_0^\alpha \right), \\ \eta(D_1^\beta) - \eta(D_2^\beta) &= 2 \left( \int_X a_0^\beta + \int_Y b_0^\beta \right). \end{aligned}$$

We recall that the integrands  $a_0^\gamma$  and  $b_0^\gamma$  with  $\gamma \in \{\alpha, \beta\}$  are independent of the flat vector bundles. Consequently, we find by subtracting the two equalities in (11.1)

$$(11.2) \quad \rho_{\alpha-\beta}(D_1) = \rho_{\alpha-\beta}(D_2).$$

A similar argument applies in the case of Cheeger-Gromov rho invariants, using Theorem 9.4 in the special case of  $D$  being the spin Dirac operator. We summarise all this in the following

**Theorem 11.2.** *Let  $(M_1, r_1 : M_1 \rightarrow B\Gamma, g_1)$  and  $(M_2, r_2 : M_2 \rightarrow B\Gamma, g_2)$  be positively bordant as in Definition (11.1). For the APS and the Cheeger-Gromov rho invariants we have*

$$(11.3) \quad \begin{aligned} \rho_{\alpha-\beta}(D_1) &= \rho_{\alpha-\beta}(D_2), \\ \rho_\Gamma(D_1) &= \rho_\Gamma(D_2). \end{aligned}$$

**11.2. Metric independence of the signature rho invariants.** We begin with a metric independence result for the signature APS invariant. Let  $\overline{M}$  be a simple edge space of depth one. We assume that  $\overline{M}$  is Witt. Let  $g_1$  and  $g_2$  be two admissible metrics on  $M$  and let  $g_\mu$  be a smooth family of admissible edge metrics on  $M$  joining  $g_1$  and  $g_2$ . We denote the signature operators associated to  $g_\mu$  by  $D_\mu$ . Up to scaling we can and we shall assume that the signature operators  $D_\mu^F$  on the links satisfy the **geometric** Witt condition (they certainly satisfy the cohomological Witt condition).

**Theorem 11.3.** *Let  $\alpha$  and  $\beta$  be any two unitary representations of  $\pi_1(\overline{M})$  in  $U(\ell)$ . Then, under the above hypothesis*

$$\rho_{\alpha-\beta}(D_1) = \rho_{\alpha-\beta}(D_2).$$

*Proof.* We shall prove that the derivative of the rho-invariant associated to  $g_\mu$  is equal to 0 on  $(0, 1)$ . Denote the flat vector bundles defined by  $\alpha$  and  $\beta$  by  $E^\alpha$  and  $E^\beta$ , respectively. Denote the operators twisted by  $E^\gamma$ ,  $\gamma \in \{\alpha, \beta\}$ , by an upper subscript  $\gamma$ . We denote the heat operators of  $(D_\mu^\alpha)^2$  and  $(D_\mu^\beta)^2$  by  $H_\mu^\alpha$  and  $H_\mu^\beta$ , respectively. Using Theorem 5.8 we find

$$(11.4) \quad T(t, \mu) := \text{Tr} (D_\mu^\alpha H_\mu^\alpha) - \text{Tr} (D_\mu^\beta H_\mu^\beta) = O(t^\infty), \text{ as } t \rightarrow 0.$$

Since the spectrum of  $(D_\mu^\alpha)^2$  and  $(D_\mu^\beta)^2$  is discrete, we also conclude  $T(t, \mu) = O(t^{-\infty})$  as  $t \rightarrow +\infty$ . Consequently, the APS rho invariant is given explicitly by evaluating the expressions (10.1) and (1.1) at  $s = 0$

$$(11.5) \quad \rho_{\alpha-\beta}(D_\mu) = \frac{1}{\sqrt{\pi}} \int_0^\infty T(t, \mu) \frac{dt}{\sqrt{t}}.$$

In order to establish a variational formula for the rho invariant, we need to ensure that the operators act on the same Hilbert space. Hence, we consider the natural isometry for a fixed  $\mu_0 \in I$  and any  $\gamma \in \{\alpha, \beta\}$

$$(11.6) \quad T_\mu^\gamma : L^2(M, {}^{ie}\Lambda^* T^* M \otimes E^\gamma, g_\mu) \rightarrow L^2(M, {}^{ie}\Lambda^* T^* M \otimes E^\gamma, g_{\mu_0}).$$

By construction, conjugating the Hodge star operator of  $(M, g_{\mu_0})$  by the isometry  $T_\mu^\gamma$  gives the Hodge star operator of  $(M, g_\mu)$ . Consequently,  $T_\mu^\gamma$  maps the domain of the twisted signature operator  $D_\mu^\gamma$  to the domain of  $D_{\mu_0}^\gamma$ . We write  $\mathcal{D}_\mu^\gamma := T_\mu^\gamma \circ D_\mu^\gamma \circ (T_\mu^\gamma)^{-1}$  with fixed domain  $\mathcal{D}(D_{\mu_0}^\gamma)$ , and  $\mathcal{H}_\mu^\gamma := T_\mu^\gamma \circ H_\mu^\gamma \circ (T_\mu^\gamma)^{-1}$ . This allows us to rewrite  $T(t, \mu)$  in terms of operators acting on a fixed Hilbert space  $L^2(M, {}^{ie}\Lambda^* T^* M \otimes E^\gamma, g_{\mu_0})$

$$(11.7) \quad T(t, \mu) = \text{Tr}(\mathcal{D}_\mu^\alpha \mathcal{H}_\mu^\alpha) - \text{Tr}(\mathcal{D}_\mu^\beta \mathcal{H}_\mu^\beta).$$

We now compute variation of  $T(t, \mu)$  in  $\mu$  using the semi-group property of the heat kernel and conclude justifying interchange of integration and differentiation by the dominated convergence theorem (we also denote derivatives in  $\mu$  evaluated at  $\mu_0$  by an upper dot)

$$(11.8) \quad \begin{aligned} \left. \frac{d}{d\mu} \right|_{\mu_0} T(t, \mu) &= \text{Tr}(\dot{\mathcal{D}}_\mu^\alpha \mathcal{H}_\mu^\alpha) - \text{Tr}(\dot{\mathcal{D}}_\mu^\beta \mathcal{H}_\mu^\beta) \\ &\quad - t \text{Tr} \left( \mathcal{D}_\mu^\alpha \left[ \left. \frac{d}{d\mu} \right|_{\mu_0} (\mathcal{D}_\mu^\alpha)^2 \right] \mathcal{H}_\mu^\alpha \right) \\ &\quad + t \text{Tr} \left( \mathcal{D}_\mu^\beta \left[ \left. \frac{d}{d\mu} \right|_{\mu_0} (\mathcal{D}_\mu^\beta)^2 \right] \mathcal{H}_\mu^\beta \right). \end{aligned}$$

Note that by construction,  $\mathcal{H}_\mu^\gamma$  maps to the domain of  $(\mathcal{D}_\mu^\gamma)^2$  that is given by  $\mathcal{D}(D_{\mu_0}^\gamma)^2$  and is in particular independent of  $\mu$ . Hence the compositions above are well-defined. Evaluating the derivatives of  $\mathcal{D}_\mu^\alpha$  and  $\mathcal{D}_\mu^\beta$  explicitly in terms of the isometries  $T_\mu^\gamma$  we find exactly as in [MuVe14, (3.7)] ( $\gamma \in \{\alpha, \beta\}$ )

$$\begin{aligned} \text{Tr}(\dot{\mathcal{D}}_\mu^\gamma \mathcal{H}_\mu^\gamma) &= \text{Tr}(\dot{T}_\mu^\gamma D_\mu^\gamma H_\mu^\gamma (T_\mu^\gamma)^{-1}) + \text{Tr}(T_\mu^\gamma \dot{D}_\mu^\gamma H_\mu^\gamma (T_\mu^\gamma)^{-1}) \\ &\quad + \text{Tr}(T_\mu^\gamma D_\mu^\gamma \dot{T}_\mu^\gamma T_\mu^\gamma H_\mu^\gamma (T_\mu^\gamma)^{-1}). \end{aligned}$$

Employing commutativity of bounded operators under the trace, we conclude

$$(11.9) \quad \text{Tr}(\dot{\mathcal{D}}_\mu^\gamma \mathcal{H}_\mu^\gamma) = \text{Tr}(\dot{D}_\mu^\gamma H_\mu^\gamma),$$

where we now view the composition  $\dot{D}_\mu^\gamma H_\mu^\gamma$  as a differential expression  $\dot{D}_\mu^\gamma$  applied to the Schwartz kernel of  $H_\mu^\gamma$ . Similar arguments prove

$$(11.10) \quad \text{Tr} \left( \mathcal{D}_\mu^\gamma \left[ \frac{d}{d\mu} \Big|_{\mu_0} (\mathcal{D}_\mu^\gamma)^2 \right] \mathcal{H}_\mu^\gamma \right) = 2 \text{Tr} \left( \dot{D}_\mu^\gamma (D_\mu^\gamma)^2 H_\mu^\gamma \right).$$

Plugging (11.9) and (11.10) into (11.8), and using the fact that by the Hodge theorem, the kernels of  $(D_\mu^\alpha)^2$  and  $(D_\mu^\beta)^2$  are independent of  $\mu$ , precluding spectral flow at zero, we obtain after justifying interchange of integration and differentiation by the dominated convergence theorem

$$\begin{aligned} \frac{d}{d\mu} \Big|_{\mu_0} \rho_{\alpha-\beta}(D_\mu) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{d}{d\mu} \Big|_{\mu_0} T(t, \mu) \frac{dt}{\sqrt{t}} \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \left( \text{Tr} \left( \dot{D}_\mu^\alpha H_\mu^\alpha \right) - \text{Tr} \left( \dot{D}_\mu^\beta H_\mu^\beta \right) \right) \frac{dt}{\sqrt{t}} \\ &\quad - \frac{2}{\sqrt{\pi}} \int_0^\infty \left( \text{Tr} \left( \dot{D}_\mu^\alpha \frac{d}{dt} H_\mu^\alpha \right) - \text{Tr} \left( \dot{D}_\mu^\beta \frac{d}{dt} H_\mu^\beta \right) \right) \sqrt{t} dt. \end{aligned}$$

Using integration by parts we obtain after cancellations

$$(11.11) \quad \frac{d}{d\mu} \Big|_{\mu_0} \rho_{\alpha-\beta}(D_\mu) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{d}{dt} \left( \sqrt{t} \text{Tr} \left( \dot{D}_\mu^\alpha H_\mu^\alpha \right) - \sqrt{t} \text{Tr} \left( \dot{D}_\mu^\beta H_\mu^\beta \right) \right) dt = 0,$$

where the last equality uses the fact that  $\text{Tr} \left( \dot{D}_\mu^\alpha H_\mu^\alpha \right) - \text{Tr} \left( \dot{D}_\mu^\beta H_\mu^\beta \right)$  is vanishing to infinite order as  $t$  goes either zero or to infinity by the Hodge theorem on Witt spaces and by Theorem 5.8.  $\square$

Next we tackle the Cheeger-Gromov rho invariant. We follow the very detailed proof of their result [CHGr85] given in Roy [Roy10] and Azzali-Wahl [AzWa11]

**Theorem 11.4.** *Under the same assumptions as in Theorem 11.3 we also have*

$$\rho_\Gamma(D_1) = \rho_\Gamma(D_2).$$

*Proof.* We continue in the notation set in the previous Theorem 11.3 and indicate the operators on the Galois covering by an upper tilde. The twisting flat vector bundle is fixed in the current setting and hence is not indicated notationally here. The Gamma-trace is denoted by  $\text{Tr}_\Gamma$ . Using Proposition 7.6 we observe

$$(11.12) \quad T_\Gamma(t, \mu) := \text{Tr}_\Gamma \left( \tilde{D}_\mu \tilde{H}_\mu \right) - \text{Tr} (D_\mu H_\mu) = O(t^\infty), \text{ as } t \rightarrow 0.$$

Due to discreteness of the spectrum of  $(D_\mu)^2$  and due to (7.10),  $t^{-1/2} T_\Gamma(t, \mu)$  is integrable in  $t \in [1, \infty)$ . Together with infinite vanishing of  $T_\Gamma(t, \mu)$  as  $t \rightarrow 0$ , we may write the Cheeger-Gromov invariant by evaluating the expressions for (10.2) at  $s = 0$

$$(11.13) \quad \rho_\Gamma(D_\mu) = \frac{1}{\sqrt{\pi}} \int_0^\infty T_\Gamma(t, \mu) \frac{dt}{\sqrt{t}}.$$

The formulae (11.8), (11.9) and (11.10) for the variation of  $T(t, \mu)$  employ only the trace class property of the operators, the fact that bounded operators commute under the

trace and the semi-group property of the heat kernel. Since these properties continue to hold under the Gamma-trace and the heat kernel on the Galois covering, we conclude by repeating the same arguments as in Theorem 11.3

$$(11.14) \quad \begin{aligned} \frac{d}{d\mu} T_\Gamma(t, \mu) &= \text{Tr}_\Gamma \left( \dot{\tilde{D}}_\mu \tilde{H}_\mu \right) - \text{Tr} \left( \dot{D}_\mu H_\mu \right) \\ &\quad - 2t \text{Tr}_\Gamma \left( \dot{\tilde{D}}_\mu (\tilde{D}_\mu)^2 \tilde{H}_\mu \right) + 2t \text{Tr} \left( \dot{D}_\mu (D_\mu)^2 H_\mu \right) \end{aligned}$$

Exactly as in (11.11) we conclude using integration by parts

$$(11.15) \quad \frac{d}{d\mu} \Big|_{\mu_0} \rho_\Gamma(D_\mu) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{d}{dt} \left( \sqrt{t} \text{Tr}_\Gamma \left( \dot{\tilde{D}}_\mu \tilde{H}_\mu \right) - \sqrt{t} \text{Tr} \left( \dot{D}_\mu H_\mu \right) \right) dt,$$

provided the limits of  $\sqrt{t} \text{Tr}_\Gamma \left( \dot{\tilde{D}}_\mu \tilde{H}_\mu \right)$  and  $\sqrt{t} \text{Tr} \left( \dot{D}_\mu H_\mu \right)$  as  $t \rightarrow 0$  and as  $t \rightarrow \infty$  are well-defined. We now study these limits. Using the Duhamel principle worked out in Proposition 7.6, the short time asymptotic behaviour of the two traces coincides and hence

$$(11.16) \quad \lim_{t \rightarrow 0} \sqrt{t} \left( \text{Tr}_\Gamma \left( \dot{\tilde{D}}_\mu \tilde{H}_\mu \right) - \text{Tr} \left( \dot{D}_\mu H_\mu \right) \right) = 0.$$

For the large time behaviour we proceed on the Galois covering as in §7.6. Note that  $\dot{\tilde{D}}_\mu \tilde{H}_\mu = \dot{*}_\mu d \tilde{H}_\mu$ , where  $*_\mu$  denotes the Hodge star operator associated to  $g_\mu$  and  $d$  is the exterior derivative. Thanks to the classical inequalities in the von Neumann algebras we can estimate

$$\begin{aligned} \left| \text{Tr}_\Gamma \left( \dot{\tilde{D}}_\mu \tilde{H}_\mu \right) \right| &= \left| \text{Tr}_\Gamma \left( \dot{*}_\mu d \tilde{H}_\mu \right) \right| \leq \| \dot{*}_\mu *_\mu^{-1} \| \text{Tr}_\Gamma \left| *_\mu d \tilde{H}_\mu \right| \\ &= \| \dot{*}_\mu *_\mu^{-1} \| \text{Tr}_\Gamma \left| \tilde{D}_\mu \tilde{H}_\mu \right| \leq C \text{Tr}_\Gamma \left| \tilde{D}_\mu \tilde{H}_\mu \right| \end{aligned}$$

for some constant  $C > 0$  independent of  $\mu$  and  $t$ . Recall the tempered measure  $m_\Gamma$  relative to  $\tilde{D}_\mu$ , introduced in §7.6. Consider the tempered measure  $n_\Gamma$  relative to  $\tilde{D}_\mu^2$ . We can then write

$$\text{Tr}_\Gamma \left| \tilde{D}_\mu \tilde{H}_\mu \right| = \int_0^\lambda \sqrt{x} e^{-tx} dn_\Gamma(x) + \int_\lambda^\infty \sqrt{x} e^{-tx} dn_\Gamma(x) =: I_1(t) + I_2(t).$$

Let us denote by  $\tilde{E}_x$  of  $\tilde{D}_\mu^2$ . Note that the function  $\sqrt{x} e^{-tx}$  has its maximum at  $x = 1/(2t)$  with value  $e^{-1/2}/\sqrt{2t}$ . Thus we can estimate  $I_1(t)$  as follows

$$\begin{aligned} I_1(t) &= \int_0^\lambda \sqrt{x} e^{-tx^2} dn_\Gamma(x) \leq (2t)^{-1/2} e^{-1/2} \int_0^\lambda dn_\Gamma(x) \\ &\leq (2t)^{-1/2} e^{-1/2} \text{Tr}_\Gamma((1 - E_0) E_\lambda (1 - E_0)). \end{aligned}$$

We estimate  $I_2(t)$  exactly as in (7.10) and obtain

$$I_2(t) \leq t^{-1/2} e^{-(t-1)\lambda} \text{Tr}_\Gamma \tilde{H}_\mu(1).$$

Note that  $\sqrt{t} I_2(t)$  converges to zero as  $t \rightarrow \infty$ . From here we conclude

$$\lim_{t \rightarrow \infty} \left| \sqrt{t} \text{Tr}_\Gamma \left( \dot{\tilde{D}}_\mu \tilde{H}_\mu \right) \right| \leq \lim_{t \rightarrow \infty} \sqrt{t} (I_1(t) + I_2(t)) \leq (2e)^{-1/2} \text{Tr}_\Gamma((1 - E_0) E_\lambda (1 - E_0)).$$

Since the left hand side of the inequality is independent of  $\lambda$ , we may take  $\lambda \rightarrow 0$ , and noting by normality of the Gamma-trace that  $\text{Tr}_\Gamma((1 - E_0)E_0(1 - E_0)) = 0$ , we finally obtain

$$(11.17) \quad \lim_{t \rightarrow \infty} \sqrt{t} \text{Tr}_\Gamma \left( \dot{\tilde{D}}_\mu \tilde{H}_\mu \right) = 0.$$

Similar argument holds for  $\sqrt{t} \text{Tr} \left( \dot{D}_\mu H_\mu \right)$  with integrals replaced by sums due to discreteness of the spectrum. In view of (11.16) and (11.17) we conclude from (11.15)

$$(11.18) \quad \left. \frac{d}{d\mu} \right|_{\mu_0} \rho_\Gamma(D_\mu) = \frac{1}{\sqrt{\pi}} \lim_{t \rightarrow \infty} \left( \sqrt{t} \text{Tr}_\Gamma \left( \dot{\tilde{D}}_\mu \tilde{H}_\mu \right) - \sqrt{t} \text{Tr} \left( \dot{D}_\mu H_\mu \right) \right) \\ - \frac{1}{\sqrt{\pi}} \lim_{t \rightarrow 0} \left( \sqrt{t} \text{Tr}_\Gamma \left( \dot{\tilde{D}}_\mu \tilde{H}_\mu \right) - \sqrt{t} \text{Tr} \left( \dot{D}_\mu H_\mu \right) \right) = 0.$$

□

### 11.3. Stratified diffeomorphism invariance of the signature rho invariants.

The metric invariance of the APS and the Cheeger-Gromov rho invariant implies immediately their stratified diffeomorphism invariance. Indeed, let  $\bar{f} : \bar{M} \rightarrow \bar{M}'$  be a stratified diffeomorphism between two edge spaces and let  $f : M \rightarrow M'$  be the induced diffeomorphism on the respective regular parts. Fix an admissible edge structure  $g'$  on  $M'$  and consider its pull-back  $f^*g'$ . This is an admissible edge structure on  $M$ ; the operation of pull-back defines a unitary isomorphism  $U : L^2(M', g) \rightarrow L^2(M, f^*g)$  and, more generally, a unitary isomorphism  $U : L^2(M', {}^{ie}\Lambda^* T^*(M')) \rightarrow L^2(M, {}^{ie}\Lambda^* T^*(M))$ . One check easily that the signature operator associated to  $f^*g'$  is obtained by conjugating the signature operator on  $M'$  through the unitary isomorphism  $U$ ; in formulae, and adopting a very precise notation for the signature operator  $D$ , we have:

$$D_M^{f^*g'} = U^{-1} \circ D_{M'}^{g'} \circ U$$

Consequently, by functional calculus,

$$D_M^{f^*g'} \exp(-t(D_M^{f^*g'})^2) = U^{-1} \circ D_{M'}^{g'} \exp(-t(D_{M'}^{g'})^2) \circ U$$

which implies the equality

$$(11.19) \quad \eta(D_M^{f^*g'}) = \eta(D_{M'}^{g'})$$

when one of the two is well defined. This implies easily that if  $\alpha'$  and  $\beta'$  are two representations of  $\pi_1(\bar{M}')$  into  $U(\ell)$  and  $\alpha := \alpha' \circ \bar{f}_*$ ,  $\beta := \beta' \circ \bar{f}_*$  then

$$(11.20) \quad \rho_{\alpha-\beta}(D_M^{f^*g'}) = \rho_{\alpha'-\beta'}(D_{M'}^{g'})$$

Since both members are metric independent we conclude that for the signature operator  $D$  the following equality holds:

$$(11.21) \quad \rho_{\alpha-\beta}(D_M) = \rho_{\alpha'-\beta'}(D_{M'})$$

A similar argument applies to two Galois  $\Gamma$ -coverings

$$\bar{M}_\Gamma \rightarrow \bar{M}, \quad \bar{M}'_\Gamma \rightarrow \bar{M}'$$

endowed with a  $\Gamma$ -equivariant stratified diffeomorphism

$$\overline{f}_\Gamma : \overline{M}_\Gamma \rightarrow \overline{M}'_\Gamma.$$

Let  $f_\Gamma : \widetilde{M} \rightarrow \widetilde{M}'$  be the induced  $\Gamma$ -equivariant diffeomorphism between the two regular strata. We fix a  $\Gamma$ -equivariant edge structure  $\tilde{g}'$  on  $\widetilde{M}'$  and we consider  $f_\Gamma^*(\tilde{g}')$  on  $\widetilde{M}$ ; we denote by  $\tilde{D}^{\tilde{g}'}$  and  $\tilde{D}^{f_\Gamma^*\tilde{g}'}$  the two corresponding signature operators. Then, proceeding as above, one proves in addition to (11.19) the following equality:

$$(11.22) \quad \eta_\Gamma(\tilde{D}^{f_\Gamma^*\tilde{g}'}) = \eta_\Gamma(\tilde{D}^{\tilde{g}'})$$

whenever one of the two is well defined. We thus conclude, by metric independence, that

$$(11.23) \quad \rho_\Gamma(\tilde{D}_{\widetilde{M}}) = \rho_\Gamma(\tilde{D}_{\widetilde{M}'}).$$

## 12. OPEN PROBLEMS AND FUTURE RESEARCH DIRECTIONS

In this final section we wish to highlight some open problems and future research directions which are strongly connected to the discussion presented here. We plan to tackle these issues in future projects.

**12.1. Iterated cone-edge singularities.** Our arguments throughout the paper apply to the case of simple incomplete edge singularities with smooth links. The case of general stratified spaces with iterated cone-edge singularities up to arbitrary depth, subject to the Witt condition is not considered here, up to a very special case of isolated cones over compact stratified spaces equipped with an exact conical metric. This contrasts for example a rather general treatment by the first named author, jointly with Albin, Leichtnam and Mazzeo e.g. in [ALMP13].

The underlying reason for that restriction is very analytic. Our arguments require a detailed understanding of the asymptotic properties of the heat kernel on the stratified space in question. While a microlocal heat kernel construction on simple edges has been performed by the second named author jointly with Mazzeo in [MAVE12], such a heat kernel description is still lacking in case of more general stratified spaces with iterated cone-edge singularities.

**12.2. Heat kernel analysis for Cheeger boundary conditions.** Throughout our arguments we have posed the geometric Witt condition to ensure essential self-adjointness of the corresponding Dirac operators. In case the geometric Witt condition is not satisfied, one may pose Cheeger boundary conditions, which have been introduced by the first named author, jointly with Albin, Leichtnam and Mazzeo in [ALMP13]. One would like to extend our discussion to the signature operator equipped with Cheeger boundary conditions; this requires a microlocal construction of the corresponding heat kernel.



**12.3. Analytic torsion on singular Galois coverings.** There has been intensive research into  $L^2$ -invariants, in particular analytic torsion on Galois coverings, cf. an incomplete list of references [BCFM05, CAMa92, LuSc99, ZHA05]. Our analysis here lays the groundwork for defining  $L^2$ -analytic torsion on Galois coverings of simple incomplete edge spaces.

**12.4. Signature formula on simple edge spaces.** The two Atiyah-Patodi-Singer index theorems established here may be applied to the special case of the signature operator in order to derive a signature formula on edge manifolds with boundary and an  $L^2$ -signature formula on  $\Gamma$ -edge manifolds with boundary (satisfying, of course, the Witt condition). As in the work of Atiyah-Patodi-Singer and in the work of Valliant [VAI08] and Lück-Schick [LuSc03], there is an additional Hodge theoretic argument in order to pass from an index formula of APS type to a true signature formula. This is ongoing work of the authors, soon to be finished.

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